# The U(n) free rigid body rigid body: Integrability and stability analysis of the equilibria



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# The $U(n)$ Free Rigid Body: Integrability and Stability Analysis of the Equilibria

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Abstract

A natural extension of the free rigid body dynamics to the unitary group $U(n)$ is considered. The dynamics is described by the Euler equation on the Lie algebra $\mathfrak{u}(n)$, which has a bi-Hamiltonian structure, and it can be reduced onto the adjoint orbits, as in the case of the $SO(n)$. The complete integrability and the stability of the isolated equilibria on the generic orbits are considered by using the method of Bolsinov and Oshemkov. In particular, it is shown that all the isolated equilibria on generic orbits are Lyapunov stable.

## 1 Introduction

The free rigid body dynamics, that is, the motion of a rigid body under no external force, is one of the typical solvable examples in theoretical mechanics. Its complete integrability and the stability properties of its equilibria have been studied and well understood since the pioneering works by Euler, Jacobi, and Poinsot. Under the influence of the theory of infinite-dimensional integrable systems, such as the Korteweg-de Vries equation, the free rigid body dynamics was generalized first to higher-dimensional rotation groups (see [25, 10, 22, 30]), then to arbitrary semi-simple Lie groups, their normal (split), compact, and normal-compact real forms (see [26, 27, 5, 6]), and finally to symmetric spaces (see the book [14]).

The goal of all these works was to prove the complete integrability of generalized free rigid body dynamics. From the viewpoint of dynamical systems theory, however, it is very natural to investigate the Lyapunov stability of relative equilibria for these dynamical systems. A well-known result for usual free rigid body dynamics in $\mathbb{R}^3$, which is a Hamiltonian system whose configuration space is the Lie group $SO(3)$, states that rotations about the long and short principal axes are Lyapunov stable, whereas rotations about the middle principal axis are unstable. For $SO(4)$, the stability of a certain class of equilibria has been studied in [13] and the complete analysis of the stability for all the equilibria was carried out in [4]. For general $SO(n)$, the stability of a

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special family has been analyzed in the Ph.D. thesis [33] and, more recently, in [18] which gives the complete analysis of the stability for generic equilibria on the basis of the paper [8].

A key feature of all these integrable systems of free rigid body type is their bi-Hamiltonian character, i.e., they are Hamiltonian with respect to two compatible Poisson structures. Bolsinov and Oshemkov [8] give a systematic method for proving the complete integrability on generic symplectic leaves of such bi-Hamiltonian systems, for describing the so-called common equilibria, i.e., the equilibria where the derivatives of all constants of motion vanish, and for giving non-degeneracy conditions of the common equilibria in the sense of Vey's and Eliasson's theorem (cf. [35, 12]). A more sophisticated description of the properties of the singularities of bi-Hamiltonian systems is given in [7]. We recall that all Hamiltonian systems on the cotangent bundle of a Lie group for a left-invariant Hamiltonian can be reduced to a Lie-Poisson system on the dual of the corresponding Lie algebra (see, e.g., [23, 31]). This Lie-Poisson Reduction Theorem also guarantees that the symplectic leaves of the dual to the Lie algebra are the connected components of the coadjoint orbits and that the restriction of the Lie-Poisson system to any coadjoint orbit is Hamiltonian relative to the orbit symplectic form and the Hamiltonian function restricted to the orbit. Since the generalized free rigid body systems are of this type, their complete integrability, as well as the stability of equilibria, is interpreted as that for the reduced system on generic coadjoint orbits. On the other hand, to think about the integrability or the stability of bi-Hamiltonian systems, it is important to suppose that the systems are defined in the real analytic category, as assumed in [8]. The generalized free rigid body dynamics are studied in the real (or complex) analytic category.

In the present paper, a natural generalization of free rigid body dynamics to the unitary group $U(n)$ is considered and the Lyapunov stability of the isolated equilibria on generic adjoint orbits is analyzed. In Section 2, we give the definition of the $U(n)$ free rigid body as a Lie-Poisson system on the dual $\mathfrak{u}(n)^*$ to the Lie algebra $\mathfrak{u}(n)$ of $U(n)$; its dynamics is described by the Euler equations. Note that this is equivalent to define the $U(n)$ free rigid body as a Hamiltonian system on $T^*U(n)$ with a $U(n)$-invariant Hamiltonian. It is also shown that the Euler equations are bi-Hamiltonian and that there is an equivalent Lax equation with parameter, just like in Manakov's formulation of the $SO(n)$ free rigid body (see [22]). In fact, the $U(n)$ free rigid body is a special case of the generalized free rigid body in [14] defined in terms of the so-called sectional operator, which is a natural Lie algebraic generalization of the classical inertia tensor. As will become apparent, the definition of the $U(n)$ free rigid body dynamics, as a dynamical system on a matrix group, is a very natural extension of the $SO(n)$ free rigid body dynamics. However, there are some subtle differences with the approach in [14]. The Lie group $U(n)$ is not semi-simple, since it has a nontrivial center, although its subgroup $SU(n)$ is a simple Lie group; thus one cannot use directly the argument in [26] to prove integrability. The relation between the free rigid body dynamics on $U(n)$ and that on $SU(n)$ is also discussed in Section 2. In fact, Mishchenko and Fomenko mentioned the restriction of the Euler equation on $\mathfrak{gl}(n, \mathbb{C})$ to $\mathfrak{u}(n)$ in [26, 27], to give an explanation of the $SO(n)$ free rigid body. However, they did not discuss this problem in great detail. As will be shown in Remark 2.3 of this section, the $U(n)$ free rigid body dynamics leaves the Lie subalgebra $\mathfrak{su}(n)$ and each level hyperplane $\left\{X \in \mathfrak{u}(n) \,\middle|\, \mathrm{Tr}(X) = \sqrt{-1}c\right\}$ of the trace function for any constant $c \in \mathbb{R}$ invariant. The restriction to $\mathfrak{su}(n) = \{X \in \mathfrak{u}(n) \,|\, \mathrm{Tr}(X) = 0\}$ can be proved to be described by an Euler equation for a Mishchenko-Fomenko $SU(n)$ free rigid body. On the other hand, the restriction to the level hyperplane $\left\{X \in \mathfrak{u}(n) \,\middle|\, \mathrm{Tr}(X) = \sqrt{-1}c\right\}$ for $c \neq 0$ is an Euler equation on $\mathfrak{su}(n)$ with respect to a Hamiltonian which is the sum of a quadratic and a nontrivial linear function, whereas the free rigid bodies discussed by Mishchenko and Fomenko [26, 27] have homogeneous quadratic Hamiltonians. Nevertheless, it is shown in this section that the inhomogeneous quadratic Hamiltonian is included in the commutative ring generated by Manakov's first integrals of Mishchenko-Fomenko $SU(n)$ free rigid body dynamics. Some of the results in this section have already been discussed in the unpublished paper [17] by Iwai.

In Section 3, the complete integrability of the $U(n)$ free rigid body dynamics is proved, using the so-called "Bolsinov-Oshemkov codimension two principle" [8]. The complete integrability of the $U(n)$ free rigid body can also be proved using the results in Section 2, since the Hamiltonian for the restriction of the $U(n)$ free rigid body to the level hyperplane $\left\{X \in \mathfrak{u}(n) \,\middle|\, \mathrm{Tr}(X) = \sqrt{-1}c\right\}$ is included in the commutative ring generated by Manakov's first integrals for the Mishchenko-Fomenko free rigid body on $\mathfrak{su}(n)$ (which is complete, as was shown in [26, 27]). However, the method based on the Bolsinov-Oshemkov codimension two principle is simpler, in the sense that it is applicable to the whole system of the $U(n)$ free rigid body without the restriction to the level hyperplanes $\left\{X \in \mathfrak{u}(n) \,\middle|\, \mathrm{Tr}(X) = \sqrt{-1}c\right\}$. Previously, in [8], the codimension two principle was applied only to bi-Hamiltonian systems on semi-simple Lie algebras. As will be mentioned at the end of Section 3, there is a third way to prove the complete integrability of the $U(n)$ free rigid body, treating it as a special case of the result by Brailov presented in [14]. Again, the Bolsinov-Oshemkov method [8] is more natural from the viewpoint of the bi-Hamiltonian structure of the Euler equation for the $U(n)$ free rigid body dynamics. In Subsection 3.1, the complete integrability of bi-Hamiltonian systems restricted to generic symplectic leaves is discussed in detail. In Subsection 3.2, the complexification of Poisson manifolds is presented. Both subsections contain detailed arguments due to their importance for the understanding of the results in [8].

In Section 4, the common equilibria, where all the derivatives of the constants of motion vanish, are described and their non-degeneracy is deduced using the result in [8].

In Section 5, the Lyapunov stability of the isolated equilibria of the $U(n)$ free rigid body is presented. The linearization of Hamilton's equations on generic adjoint orbits around the common equilibria is carried out. As opposed to the stability analysis for the $SO(n)$ free rigid body [4, 18], these equilibria are all linearly stable. From the linear stability of these equilibria, one can also conclude their Lyapunov stability by using the results in the previous sections and Vey's theorem [35]. This result is remarkable, since it shows that the stability analysis for the $U(n)$ free rigid body is considerably simpler than that of the $SO(n)$ free rigid body, even in low dimensions. It should be mentioned that the stability property of the common equilibria can also be shown by another algebro-geometric method recently proposed in [18]. The advantage of the method in the present paper is, however, that the linearization of Hamilton's equation and the frequencies of the system around the equilibria are explicitly computed.

In the final section, the special case of the $U(2)$ free rigid body is discussed as an example.

# 2 $U(n)$ free rigid body

We begin with some basic notations. As usual, the real Lie group consisting of all $n \times n$ complex unitary matrices is denoted by $U(n)$. Its Lie algebra $\mathfrak{u}(n)$ is the set of all $n \times n$ skew-Hermitian matrices equipped with the standard commutator $[\cdot,\cdot]$ of matrices. This Lie algebra has an invariant inner product $\langle\cdot,\cdot\rangle$ defined by $\langle X, Y\rangle := \mathrm{Tr}\left(X^*Y\right) = -\mathrm{Tr}\left(XY\right)$, for all $X, Y \in \mathfrak{u}(n)$, which is unique up to a scalar multiple. Here, $X^*$ denotes the Hermitian conjugate of $X$, i.e., $X^* := \overline{X}^{\mathrm{T}}$. Invariance of $\langle\cdot,\cdot\rangle$ means that

$$\langle X, [Y, Z]\rangle = \langle [X, Y], Z\rangle, \quad \text{for all} \quad X, Y, Z \in \mathfrak{u}(n). \tag{2.1}$$

By means of this inner product, the Lie algebra $\mathfrak{u}(n)$ can be identified with its dual $\mathfrak{u}(n)^*$; we implement this identification in the rest of the paper. The vector space $\mathfrak{u}(n)^* = \mathfrak{u}(n)$, is a Lie-Poisson space relative to the Lie-Poisson bracket

$$\{F, G\}(X) = \langle X, [\nabla F(X), \nabla G(X)]\rangle, \quad \text{for all} \quad F, G \in \mathcal{C}^{\omega}\left(\mathfrak{u}(n)^*\right), \quad X \in \mathfrak{u}(n)^* = \mathfrak{u}(n), \tag{2.2}$$

where $\mathcal{C}^{\omega}\left(\mathfrak{u}(n)^*\right)$ denotes the ring of real analytic functions on $\mathfrak{u}(n)^* = \mathfrak{u}(n)$ and the gradients $\nabla F(X), \nabla G(X)$ with respect to $\langle\cdot,\cdot\rangle$ are defined in the following manner: if $\mathsf{d}$ denotes the exterior (or, in this case, the standard) derivative of a smooth real valued function on $\mathfrak{u}(n)$, set

$$\mathsf{d}F(Y)\cdot Y = \langle Y, \nabla F(Y)\rangle\,,$$

for every $Y \in \mathfrak{u}(n)$.

**Remark 2.1.** The Lie-Poisson bracket (2.2) naturally extends to the algebra $\mathcal{C}^\infty\left(\mathfrak{u}(n)^*\right)$ of infinitely many differentiable functions on $\mathfrak{u}(n)^*$. However, we focus on the case of real analytic functions, since this hypothesis is needed in the proof of integrability of the $U(n)$ free rigid body dynamics restricted to generic symplectic leaves; this is done in Section 3 and is based on the result in Proposition 3.11. $\Diamond$

Let $\Xi_F$ denote the Lie-Poisson Hamiltonian vector field on $\mathfrak{u}(n)$ defined by $F \in \mathcal{C}^\omega\left(\mathfrak{u}(n)\right)$, i.e., $\mathsf{d}G(Y)\cdot \Xi_F(Y) = \{F, G\}\,(Y)$, for any $G \in \mathcal{C}^\omega\left(\mathfrak{u}(n)\right)$ and $Y \in \mathfrak{u}(n)$. Then $\langle \Xi_F(Y), \nabla G(Y)\rangle = \mathsf{d}G(Y)\cdot \Xi_F(Y) = \{F, G\}\,(Y) = \langle Y, [\nabla F(Y), \nabla G(Y)]\rangle = \langle [Y, \nabla F(Y)]\,, \nabla G(Y)\rangle$, which yields the general formula

$$\Xi_F(Y) = [Y, \nabla F(Y)], \quad Y \in \mathfrak{u}(n). \tag{2.3}$$

## 2.1 The $U(n)$ free rigid body dynamics as a Lie-Poisson system

The natural analogue of the inertia tensor for the $SO(n)$ free rigid body, is the *moment of inertia operator*, the linear mapping $\mathcal{J} : \mathfrak{u}(n) \ni X \mapsto \mathsf{J}X + X\mathsf{J} \in \mathfrak{u}(n)$, associated to an arbitrary, but fixed, $n\times n$ Hermitian matrix $\mathsf{J}$; $\mathcal{J}$ is symmetric with respect to $\langle\cdot,\cdot\rangle$. We assume that $\mathcal{J}$ is positive-definite, as in the case of the ordinary free rigid body dynamics, and let $H(X) := \dfrac{1}{2}\left\langle X, \mathcal{J}^{-1}(X)\right\rangle$, $X \in \mathfrak{u}(n) \cong \mathfrak{u}(n)^*$, be the kinetic (and hence total) energy of the $U(n)$ rigid body.

**Definition 2.1.** *The Lie-Poisson system* $\left(\mathfrak{u}(n)^* = \mathfrak{u}(n), \{\cdot,\cdot\}\,, H\right)$ *is called the* $U(n)$ free rigid body*.*

Since $\nabla H(X) = \mathcal{J}^{-1}(X)$, its associated Hamilton equation (2.3) has the form

$$\frac{\mathsf{d}}{\mathsf{d}t}X = \left[X, \mathcal{J}^{-1}\left(X\right)\right], \quad X \in \mathfrak{u}(n), \tag{2.4}$$

which is also called the *Euler equation* on $\mathfrak{u}(n)$.

There are several important consequences of the definition of the $U(n)$ free rigid body as a Lie-Poisson system (see, e.g., [1], [14], [23], [24], [30]). First, the Hamiltonian vector field (2.4) is necessarily tangent to each (co)adjoint orbit $\mathcal{O}$ of $U(n)$ in $\mathfrak{u}(n)$. Every orbit $\mathcal{O}$ is a symplectic manifold relative to the *orbit* (or Kirillov-Kostant-Souriau) *symplectic form*

$$\omega(\xi)\left(\mathrm{ad}^*_X\xi, \mathrm{ad}^*_Y\xi\right) := \langle \xi, [X, Y]\rangle\,, \tag{2.5}$$

where $\xi \in \mathcal{O}$, $X, Y \in \mathfrak{u}(n)$, $\mathrm{ad}_X Z := [X, Z]$ for any $Z \in \mathfrak{u}(n)$, and $\mathrm{ad}^*_X = -\mathrm{ad}_X : \mathfrak{u}(n)^* \equiv \mathfrak{u}(n) \to \mathfrak{u}(n)$ is the dual operator of $\mathrm{ad}_X$. In the formula above we have used the fact that $T_\xi\mathcal{O} = \{\mathrm{ad}^*_X\xi \mid X \in \mathfrak{u}(n)\}$ for $\xi \in \mathcal{O}$. The Hamiltonian vector field on $\mathcal{O}$ defined by $H|_{\mathcal{O}}$ and the symplectic form (2.5) coincides with (2.4). Since $U(n)$ is compact and connected, each adjoint orbit $\mathcal{O}$ is a compact connected submanifold of $\mathfrak{u}(n)$; the coadjoint orbits are the symplectic leaves of the Lie-Poisson space $\mathfrak{u}(n)^* = \mathfrak{u}(n)$. Second, the cotangent bundle $T^*U(n)$ is diffeomorphic to $U(n) \times \mathfrak{u}(n)^*$ by left-translation: $T^*U(n) \ni (g, \alpha_g) \mapsto (g, T^*_e L_g(\alpha_g)) \in U(n) \times \mathfrak{u}(n)^*$, where $g \in U(n)$, $\alpha_g \in T^*_g U(n)$, $L_g$ denotes left-translation by $g \in U(n)$, i.e., $h \mapsto gh$ for all $h \in U(n)$, $T_e L_g : T_e U(n) = \mathfrak{u}(n) \to T_g U(n)$ is the tangent map (derivative) of $L_g$, and $e \in U(n)$ is the unit element of the group (the identity matrix). Using this identification, the Hamiltonian $H$ on $\mathfrak{u}(n)^*$ induces a left-invariant function $\widetilde{H}$ on $T^*U(n)$ given by $\widetilde{H}\left(g, \alpha_g\right) := H\left(T^*_e L_g(\alpha_g)\right)$, $(g, \alpha_g) \in$

$T^*U(n)$. By the Lie-Poisson Reduction Theorem (see, e.g., [23, §13.1] or [31, §6.1]), it follows that the Lie-Poisson system $\left(\mathfrak{u}(n)^*, \{\cdot,\cdot\}, H\right)$ is the reduced system of the Hamiltonian system $\left(T^*U(n), \Omega, \widetilde{H}\right)$, where $\Omega$ is the canonical cotangent bundle symplectic form, using the momentum mapping $T^*U(n) \cong U(n) \times \mathfrak{u}(n)^* \ni (g,\eta) \mapsto \mathrm{Ad}^*_{g^{-1}}\eta \in \mathfrak{u}(n)^*$ of right translation. Further, if one uses the Marsden-Weinstein reduction theorem [24], the Hamiltonian system $\left(T^*U(n), \Omega, \widetilde{H}\right)$ is reduced to the system on coadjoint orbits $\mathcal{O}$ equipped with the orbit symplectic form and the restricted Hamiltonian $H|_{\mathcal{O}}$.

The goals of this paper are the proof of the complete integrability of the $U(n)$ free rigid body (2.4) and the study of the Lyapunov stability of its equilibria on generic (co)adjoint orbits $\mathcal{O}$.

**Remark 2.2.** In the definition of the $U(n)$ free rigid body, we may assume, without loss of generality, that the Hermitian matrix $\mathsf{J}$ is real diagonal. This is guaranteed by the following transformation formula of the Euler equation, the proof of which is straightforward. $\diamondsuit$

**Lemma 2.1.** *Let $g \in U(n)$ and $\mathcal{J}_g(X) := \mathsf{J}^g X + X\mathsf{J}^g$, $\mathsf{J}^g := g\mathsf{J}g^{-1}$. Then, the Euler equation* (2.4) *is transformed into*

$$\frac{\mathsf{d}}{\mathsf{d}t}\mathrm{Ad}_g X = \left[\mathrm{Ad}_g X, \mathcal{J}_g{}^{-1}\left(\mathrm{Ad}_g X\right)\right].$$

## 2.2 Bi-Hamiltonian structures

One of the important features of the Euler equation for the $U(n)$ free rigid body dynamics is its bi-Hamiltonian character. We briefly recall below the basic notions related to bi-Hamiltonian structures needed in this paper.

Let $M$ be a smooth manifold and $\{\cdot,\cdot\}_0$ a Poisson bracket on $\mathcal{C}^\infty(M)$[1]. Denote by $\mathcal{A}_0$ the skew-symmetric contravariant two-tensor of type $(2,0)$ defined by the bracket $\{\cdot,\cdot\}_0$, i.e., $\{f,g\}_0(x) = \mathcal{A}_0(x)\left(\mathsf{d}f(x), \mathsf{d}g(x)\right)$, for all $f,g \in \mathcal{C}^\infty(M)$ and all $x \in M$. The functions $f,g \in \mathcal{C}^\infty(M)$ are said to be in *involution*, if $\{f,g\}_0 = 0$. More generally, a subset $\mathcal{F} \subset \mathcal{C}^\infty(M)$ is called *involutive*, if all its elements are in involution.

Let $\{\cdot,\cdot\}_1$ be another Poisson bracket on $\mathcal{C}^\infty(M)$ with associated tensor $\mathcal{A}_1$. The Poisson tensors $\mathcal{A}_0$ and $\mathcal{A}_1$ are said to be *compatible* if their sum $\mathcal{A}_0 + \mathcal{A}_1$ is also a Poisson tensor. Clearly, this is equivalent to the fact that any linear combination $\lambda_1\mathcal{A}_0 + \lambda_1\mathcal{A}_1$, $\lambda_0, \lambda_1 \in \mathbb{R}$, is a Poisson tensor on $M$. As in [8], we denote by $\mathcal{P} := \{\lambda_0\mathcal{A}_0 + \lambda_1\mathcal{A}_1 \mid (\lambda_0 : \lambda_1) \in P_1(\mathbb{R})\}$ the pencil of Poisson tensors spanned by $\mathcal{A}_0$ and $\mathcal{A}_1$. (In the complex case, one considers the complex pencil $\mathcal{P} = \{\lambda_0\mathcal{A}_0 + \lambda_1\mathcal{A}_1 \,|(\lambda_0 : \lambda_1) \in P_1(\mathbb{C})\}$. )

The *rank* of a Poisson structure is defined to be the maximal rank of the associated skew-symmetric tensor field of type $(2,0)$, regarded as a skew-symmetric bilinear map on each cotangent vector space $T^*_xM$ to the Poisson manifold $M$. The *rank of the Poisson pencil* $\mathcal{P}$ is defined by

$$\mathrm{rank}\,(\mathcal{P}) := \max_{\lambda\in\mathbb{R}} \mathrm{rank}\,(\mathcal{A}_\lambda)\,, \quad \text{where} \quad \mathcal{A}_\lambda := \mathcal{A}_0 + \lambda\mathcal{A}_1 \in \mathcal{P}.$$

The Poisson structure defined by $\mathcal{A}_\lambda \in \mathcal{P}$ is called *generic* if $\mathrm{rank}\,\mathcal{A}_\lambda = \mathrm{rank}\,\mathcal{P}$, i.e., $\mathrm{rank}\,\mathcal{A}_\lambda$ is maximal.

Let $\mathcal{F}_\mathcal{P}$ be the commutative ring of functions generated, with respect to the usual multiplication of functions, by the Casimir functions of all the generic Poisson brackets $\{\cdot,\cdot\}_\lambda$ corresponding to

[1]One could take the class of functions to be real (or complex) analytic or polynomial functions on $M$, if $M$ is a real (or complex) analytic manifold or an affine algebraic variety. In fact, in the study of the $U(n)$ free rigid body, we will work with the Lie-Poisson structure (2.2) on $\mathcal{C}^\omega(\mathfrak{u}(n))$. The notion of Poisson manifold, as well as that of bi-Hamiltonian structures, can be naturally extended to the case of complex manifolds, which is presented in Subsection 3.2.

the Poisson tensors $\mathcal{A}_\lambda = \mathcal{A}_0 + \lambda\mathcal{A}_1 \in \mathcal{P}$, namely by those functions $f \in \mathcal{C}^\infty(M)$ for which there exists at least one $\lambda \in P_1(\mathbb{R})$ such that $\{f, g\}_\lambda = 0$ for all $g \in \mathcal{C}^\infty(M)$. By [8, Proposition 1], $\mathcal{F}_\mathcal{P}$ is also commutative with respect to all the Poisson brackets in $\mathcal{P}$.

Usually, one wants to prove that a given Hamiltonian system on $(M, \mathcal{A}_0)$ is integrable, which means that it is completely integrable in the classical Liouville sense on all maximal dimensional symplectic leaves. Concretely, this means that if $\mathcal{L}$ is such a symplectic leaf, then one needs to show that there are $\frac{1}{2}\dim\mathcal{L}$ functionally independent first integrals in $\mathcal{F}_\mathcal{P}|_\mathcal{L}$, i.e., their differentials are linearly independent almost everywhere on $\mathcal{L}$. The key idea to the complete integrability on such a symplectic leaf is the *completeness* of a set of involutive functions with respect to the given Poisson structure, which we discuss in detail in Subsection 3.1.

## 2.3 Bi-Hamiltonian property of the $\mathfrak{u}(n)$-Euler equation

As in the case of the $SO(n)$ free rigid body dynamics, the Euler equation (2.4) is also a Hamiltonian system with respect to another Poisson bracket than the standard Lie-Poisson bracket $\{\cdot, \cdot\}$ given in (2.2). This is an easy direct verification, as for the $SO(n)$ case given in [28]. The bi-Hamiltonian structure of free rigid body dynamics on $SO(n)$ and on more general semi-simple Lie algebras was found in [5, 6]. In order to find the bi-Hamiltonian structure of the Euler equation (2.4) for the $U(n)$ free rigid body dynamics, we consider the operation $[X, Y]_\mathsf{A} := X\mathsf{A}Y - Y\mathsf{A}X$, where $X, Y \in \mathfrak{u}(n)$ and $\mathsf{A}$ is a fixed $n \times n$ Hermitian matrix. It is straightforward to check that $[\cdot, \cdot]_\mathsf{A}$ is a Lie bracket and we denote by $\mathfrak{u}(n)_\mathsf{A}$ the real vector space underlying the usual Lie algebra $\mathfrak{u}(n)$ endowed with the Lie bracket $[\cdot, \cdot]_\mathsf{A}$; in particular, the Lie algebra $\mathfrak{u}(n) = \mathfrak{u}(n)_\mathsf{E}$, where $\mathsf{E}$ is the identity matrix. As before, we use the inner product $\langle X, Y\rangle := \mathrm{Tr}\,(X^*Y) = -\mathrm{Tr}\,(XY)$, for all $X, Y \in \mathfrak{u}(n)$, to identify the real vector space $\mathfrak{u}(n)$ with its dual $\mathfrak{u}(n)^*$. Thus, the Lie-Poisson bracket $\{\cdot, \cdot\}_\mathsf{A}$ on $\mathfrak{u}(n)_\mathsf{A}{}^* \equiv \mathfrak{u}(n)_\mathsf{A}$ has the expression

$$\{F, G\}_\mathsf{A}(X) := \langle X, [\nabla F(X), \nabla G(X)]_\mathsf{A}\rangle, \quad \text{for all} \quad F, G \in \mathcal{C}^\infty(\mathfrak{u}(n)), \quad X \in \mathfrak{u}(n). \tag{2.6}$$

Since $\lambda_0 [\cdot, \cdot]_{\mathsf{A}_0} + \lambda_1 [\cdot, \cdot]_{\mathsf{A}_1} = [\cdot, \cdot]_{\lambda_0\mathsf{A}_0 + \lambda_1\mathsf{A}_1}$ is a Lie bracket for any $\lambda_0, \lambda_1 \in \mathbb{R}$ and fixed Hermitian matrices $\mathsf{A}_0, \mathsf{A}_1$, the Lie-Poisson brackets $\{\cdot, \cdot\}_{\mathsf{A}_0}$ and $\{\cdot, \cdot\}_{\mathsf{A}_1}$ are compatible. We denote by $\Xi_F^{(\mathsf{A})}$ the Hamiltonian vector field of $F \in \mathcal{C}^\omega(\mathfrak{u}(n))$ relative to the Poisson bracket (2.6). Since, for any $G \in \mathcal{C}^\omega(\mathfrak{u}(n))$ and $Y \in \mathfrak{u}(n)$, we have

$$\begin{aligned}
\left\langle \Xi_F^{(\mathsf{A})}(Y), \nabla G(Y) \right\rangle &= \mathsf{d}G(Y) \cdot \Xi_F^{(\mathsf{A})}(Y) = \{F, G\}_\mathsf{A}(Y) = \langle Y, [\nabla F(Y), \nabla G(Y)]_\mathsf{A}\rangle \\
&= -\mathrm{Tr}\,(Y\nabla F(Y)\mathsf{A}\nabla G(Y) - Y\nabla G(Y)\mathsf{A}\nabla F(Y)) \\
&= -\mathrm{Tr}\,((Y\nabla F(Y)\mathsf{A} - \mathsf{A}\nabla F(Y)Y)\nabla G(Y)) \\
&= \langle Y\nabla F(Y)\mathsf{A} - \mathsf{A}\nabla F(Y)Y, \nabla G(Y)\rangle,
\end{aligned}$$

we get

$$\Xi_F^{(\mathsf{A})}(Y) = Y\nabla F(Y)\mathsf{A} - \mathsf{A}\nabla F(Y)Y. \tag{2.7}$$

Note that the inner product $\langle \cdot, \cdot\rangle$ is not invariant for the Lie bracket $[\cdot, \cdot]_\mathsf{A}$, i.e., identity (2.1) does not hold when replacing $[\cdot, \cdot]$ with $[\cdot, \cdot]_\mathsf{A}$.

**Proposition 2.2.** *The Euler equation* (2.4) *for the* $U(n)$ *free rigid body dynamics is Hamiltonian for the Lie-Poisson system* $\left(\mathfrak{u}(n)^*_{\mathsf{J}^2}, \{\cdot, \cdot\}_{\mathsf{J}^2}, H'\right)$, *where* $H'(X) := \dfrac{1}{2}\mathrm{Tr}\left(\mathsf{J}^{-1}X\mathsf{J}^{-1}\mathcal{J}^{-1}(X)\right)$, $X \in \mathfrak{u}(n)$. *Hence, the Euler equation* (2.4) *for the* $U(n)$ *free rigid body is bi-Hamiltonian.*

*Proof.* Since for arbitrary skew-Hermitian $n \times n$ matrices $X, Y$ we have

$$\begin{aligned}\mathsf{d}H'(Y) \cdot X &= \frac{1}{2}\mathrm{Tr}\left(\mathsf{J}^{-1}X\mathsf{J}^{-1}\mathcal{J}^{-1}(Y)\right) + \frac{1}{2}\mathrm{Tr}\left(\mathsf{J}^{-1}Y\mathsf{J}^{-1}\mathcal{J}^{-1}(X)\right) \\ &= \frac{1}{2}\mathrm{Tr}\left(X\mathsf{J}^{-1}\mathcal{J}^{-1}(Y)\mathsf{J}^{-1}\right) + \frac{1}{2}\mathrm{Tr}\left(\mathcal{J}^{-1}\left(\mathsf{J}^{-1}Y\mathsf{J}^{-1}\right)X\right),\end{aligned}$$

it follows that

$$\nabla H'(Y) = -\frac{1}{2}\mathsf{J}^{-1}\mathcal{J}^{-1}(Y)\mathsf{J}^{-1} - \frac{1}{2}\mathcal{J}^{-1}\left(\mathsf{J}^{-1}Y\mathsf{J}^{-1}\right) = -\mathsf{J}^{-1}\mathcal{J}^{-1}(Y)\mathsf{J}^{-1},$$

where we used the identity $\mathcal{J}^{-1}\left(\mathsf{J}^{-1}Y\mathsf{J}^{-1}\right) = \mathsf{J}^{-1}\mathcal{J}^{-1}(Y)\mathsf{J}^{-1}$. Thus, by (2.7) we get

$$\begin{aligned}\Xi_{H'}^{(\mathsf{J}^2)}(Y) &= Y\nabla H'(Y)\mathsf{J}^2 - \mathsf{J}^2\nabla H'(Y)Y = -Y\mathsf{J}^{-1}\mathcal{J}^{-1}(Y)\mathsf{J}^{-1}\mathsf{J}^2 + \mathsf{J}^2\mathsf{J}^{-1}\mathcal{J}^{-1}(Y)\mathsf{J}^{-1}Y \\ &= -\left[\left(\mathcal{J}^{-1}(Y)\right)^2, \mathsf{J}\right],\end{aligned} \tag{2.8}$$

as a direct verification shows. Since $\left[Y, \mathcal{J}^{-1}(Y)\right] = \left[\mathsf{J}, \left(\mathcal{J}^{-1}(Y)\right)^2\right]$, the proposition is proved. □

## 2.4 Manakov equation and involution of the integrals of motion

In a similar manner to the Manakov equation for the $SO(n)$ free rigid body [22], one can find a Lax equation with a complex parameter which is equivalent to the Euler equation (2.4). Indeed, since $\left[\mathsf{J}^2, \mathcal{J}^{-1}(X)\right] + [X, \mathsf{J}] = 0$, it follows that the equation

$$\frac{\mathsf{d}}{\mathsf{d}t}\left(\sqrt{-1}X + \lambda\mathsf{J}^2\right) = \left[\sqrt{-1}X + \lambda\mathsf{J}^2, \mathcal{J}^{-1}(X) - \sqrt{-1}\lambda\mathsf{J}\right], \tag{2.9}$$

where $\lambda$ is a time-independent complex parameter, is equivalent to the Euler equation (2.4) (this is the same argument as the one in [22]).

By the standard argument for Lax equations with a parameter (see, e.g., [1, §5.5.7], [3, Introduction]), it follows that the eigenvalues of the matrix $\sqrt{-1}X + \lambda\mathsf{J}$ are constants of motion for the Euler equation, i.e., the flow of (2.9) is *isospectral*. Equivalently, the functions $f_k(X) := \frac{1}{k}\mathrm{Tr}\left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^k$, $k = 1, \ldots, n$, are first integrals. If $\lambda$ is a real number, the matrix $\left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^k$ is Hermitian due to the factor $\sqrt{-1}$, so that $f_k$ is real valued for $\lambda \in \mathbb{R}$. Since $\lambda$ is time independent, the coefficients $I_j^{(k)}(X)$ of the polynomial expansion

$$f_k(X) = \sum_{j=0}^{k}\lambda^j I_j^{(k)}(X) \tag{2.10}$$

for $f_k$ in $\lambda$ are also first integrals. Note that the coefficients $I_k^{(k)}(X) = \frac{1}{k}\mathrm{Tr}\left(\mathsf{J}^{2k}\right)$ are constants and $I_0^{(k)}(X) = \frac{\left(\sqrt{-1}\right)^k}{k}\mathrm{Tr}\left(X^k\right)$, $k = 1, \ldots, n$, are Casimir functions for the Poisson bracket $\{\cdot, \cdot\}$. The functions $I_j^{(k)}$, $k = 1, \ldots, n$, $j = 1, \ldots, k-1$, form a set of $\frac{1}{2}n(n-1)$ first integrals, the number of which is equal to half the dimension of the generic (co)adjoint orbits in $\mathfrak{u}(n)$.

Next, we prove Poisson commutativity of the family $\left\{ I_j^{(k)} \,\middle|\, k = 1, \ldots, n,\ j = 1, \ldots, k-1 \right\}$. Although this is standard (and explicitly shown in [17]), we give the proof for the sake of completeness. The first key relation is a link between the gradients of the functions $I_j^{(k)}$ and the coefficients

$C_j^{(k)}(X)$ of $\lambda^j$ in the expansion

$$\left(\sqrt{-1}X + \lambda \mathsf{J}\right)^k = \sum_{j=0}^{k} \lambda^j C_j^{(k)}(X).$$

We have

$$\nabla I_j^{(k)}(X) = -\sqrt{-1}C_j^{(k-1)}(X), \tag{2.11}$$

since

$$\begin{aligned}\sum_{j=1}^{k} \lambda^j \left\langle \mathsf{d}X, \nabla I_j^{(k)}(X)\right\rangle &= \langle \mathsf{d}X, \nabla f_k(X)\rangle = \mathsf{d}f_k(X) \\ &= \sqrt{-1}\mathrm{Tr}\left(\left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^{k-1}\mathsf{d}X\right) = \sqrt{-1}\sum_{j=0}^{k-1}\lambda^j \mathrm{Tr}\left(C_j^{k-1}(X)\mathsf{d}X\right) \\ &= \sum_{j=0}^{k-1}\lambda^j \left\langle \mathsf{d}X, -\sqrt{-1}C_j^{k-1}(X)\right\rangle.\end{aligned}$$

The second key identity is the following:

$$\sqrt{-1}\left[X, C_j^{(k)}(X)\right] + \left[\mathsf{J}^2, C_{j-1}^{(k)}(X)\right] = 0, \quad k = 1, \ldots, n, \quad j = 1, \ldots, k. \tag{2.12}$$

Indeed,

$$\begin{aligned}0 &= \left[\sqrt{-1}X + \lambda\mathsf{J}^2, \left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^k\right] = \left[\sqrt{-1}X + \lambda\mathsf{J}^2, \sum_{j=0}^{k}\lambda^j C_j^{(k)}(X)\right] \\ &= \sum_{j=0}^{k}\lambda^j\sqrt{-1}\left[X, C_j^{(k)}(X)\right] + \sum_{j=1}^{k+1}\lambda^j\left[\mathsf{J}^2, C_{j-1}^{(k)}(X)\right].\end{aligned}$$

**Theorem 2.3.** *The functions* $\left\{ I_j^{(k)} \,\middle|\, k = 1, \ldots, n,\ j = 1, \ldots, k-1\right\}$ *are in involution with respect to all Poisson brackets* $\{\cdot,\cdot\}_{\mathsf{E}+\lambda\mathsf{J}^2}$ *on* $\mathfrak{u}(n)$ *in the pencil generated by the Lie-Poisson bracket* $\{\cdot,\cdot\}$ *and the Poisson bracket* $\{\cdot,\cdot\}_{\mathsf{J}^2}$.

*Proof.* We use an argument similar to the one in [28].
Step 1: Involution of the functions $\left\{ I_j^{(k)} \,\middle|\, k = 1, \ldots, n,\ j = 1, \ldots, k-1\right\}$ with respect to the Lie-Poisson bracket $\{\cdot,\cdot\}$. This follows by applying iteratively the identity

$$\left\{I_\alpha^{(k)}, I_\beta^{(l)}\right\} = \left\{I_{\alpha-1}^{(k)}, I_{\beta+1}^{(l)}\right\}$$

for all $k, l = 1, \ldots, n$, $\alpha = 1, \ldots, k$, and $\beta = 0, \ldots, k-1$, since $I_k^{(k)}$ is constant and $I_0^{(k)}$ is a Casimir function. We prove now this identity.

$$\begin{aligned}
\left\{I_\alpha^{(k)}, I_\beta^{(l)}\right\}(X) &\overset{(2.2)}{=} \left\langle X, \left[\nabla I_\alpha^{(k)}(X), \nabla I_\beta^{(l)}(X)\right]\right\rangle \\
&\overset{(2.11)}{=} \left\langle X, -\left[C_\alpha^{(k-1)}(X), C_\beta^{(l-1)}(X)\right]\right\rangle \\
&\overset{(2.1)}{=} \left\langle -\left[X, C_\alpha^{(k-1)}(X)\right], C_\beta^{(l-1)}(X)\right\rangle \\
&\overset{(2.12)}{=} \left\langle -\sqrt{-1}\left[\mathsf{J}^2, C_{\alpha-1}^{(k-1)}(X)\right], C_\beta^{(l-1)}(X)\right\rangle \\
&= \left\langle \left[C_{\alpha-1}^{(k-1)}(X), \sqrt{-1}\mathsf{J}^2\right], C_\beta^{(l-1)}(X)\right\rangle \\
&\overset{(2.1)}{=} \left\langle C_{\alpha-1}^{(k-1)}(X), \left[\sqrt{-1}\mathsf{J}^2, C_\beta^{(l-1)}(X)\right]\right\rangle \\
&\overset{(2.12)}{=} \left\langle C_{\alpha-1}^{(k-1)}(X), \left[X, C_{\beta+1}^{(l-1)}(X)\right]\right\rangle \\
&\overset{(2.1)}{=} -\left\langle X, \left[C_{\alpha-1}^{(k-1)}(X), C_{\beta+1}^{(l-1)}(X)\right]\right\rangle \\
&\overset{(2.11)}{=} \left\langle X, \left[\nabla I_{\alpha-1}^{k}(X), \nabla I_{\beta+1}^{l}(X)\right]\right\rangle \\
&\overset{(2.2)}{=} \left\{I_{\alpha-1}^{(k)}, I_{\beta+1}^{(l)}\right\}(X).
\end{aligned}$$

This proves the involution of the integrals relative to the Lie-Poisson bracket (2.2).

Step 2. If $f_k(X) = \dfrac{1}{k}\mathrm{Tr}\left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^k$ and $G \in \mathcal{C}^\omega(\mathfrak{u}(n))$ are arbitrary, we have

$$\{f_k, G\} = \lambda\{f_{k-1}, G\}_{\mathsf{J}^2}, \qquad k = 1, 2, 3, \ldots. \tag{2.13}$$

If $k = 1$, the identity (2.13) is obvious, because $\sqrt{-1}\mathrm{Tr}(X)$ is a Casimir function of the Lie-Poisson bracket $\{\cdot, \cdot\}$. So, assume $k \geq 2$. Since $\nabla f_k(X) = \sqrt{-1}\left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^{k-1}$, we get

$$\begin{aligned}
\{f_k, G\}(X) &\overset{(2.2)}{=} \langle X, [\nabla f_k(X), \nabla G(X)]\rangle \\
&= \left\langle X, \left[\sqrt{-1}\left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^{k-1}, \nabla G(X)\right]\right\rangle \\
&\overset{(2.1)}{=} \left\langle \left[\sqrt{-1}X, \left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^{k-1}\right], \nabla G(X)\right\rangle.
\end{aligned}$$

Now, as in the proof of (2.12), we have

$$\begin{aligned}
\left[\sqrt{-1}X, \left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^{k-1}\right] &= -\lambda\left[\mathsf{J}^2, \left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^{k-1}\right] \\
&= -\lambda\left(\sqrt{-1}\left[\mathsf{J}^2, \left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^{k-2}X\right] + \lambda\left[\mathsf{J}^2, \left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^{k-2}\mathsf{J}^2\right]\right) \\
&= -\lambda\left(\sqrt{-1}\left[\mathsf{J}^2, \left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^{k-2}X\right] + \lambda\left[\mathsf{J}^2, \left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^{k-2}\right]\mathsf{J}^2\right) \\
&= -\lambda\left(\sqrt{-1}\left[\mathsf{J}^2, \left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^{k-2}X\right] - \sqrt{-1}\left[X, \left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^{k-2}\right]\mathsf{J}^2\right) \\
&= -\lambda\sqrt{-1}\left(\mathsf{J}^2\left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^{k-2}X - \left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^{k-2}X\mathsf{J}^2\right. \\
&\qquad \left. - X\left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^{k-2}\mathsf{J}^2 + \left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^{k-2}X\mathsf{J}^2\right) \\
&= -\lambda\sqrt{-1}\left(\mathsf{J}^2\left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^{k-2}X - X\left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^{k-2}\mathsf{J}^2\right),
\end{aligned}$$

and hence

$$\begin{aligned}
\{f_k, G\}(X) &= \lambda\sqrt{-1}\left\langle X\left(\sqrt{-1}X + \lambda \mathsf{J}^2\right)^{k-2}\mathsf{J}^2 - \mathsf{J}^2\left(\sqrt{-1}X + \lambda \mathsf{J}^2\right)^{k-2}X, \nabla G(X)\right\rangle \\
&= \lambda\left\langle X\nabla f_{k-1}(X)\mathsf{J}^2 - \mathsf{J}^2\nabla f_{k-1}(X)X, \nabla G(X)\right\rangle \\
&\overset{(2.7)}{=} \lambda\left\langle \Xi^{(\mathsf{J}^2)}_{f_{k-1}}(X), \nabla G(X)\right\rangle \\
&\overset{(2.6)}{=} \lambda\left\{f_{k-1}, G\right\}_{\mathsf{J}^2}(X),
\end{aligned}$$

which proves (2.13).

Step 3. Involution of the functions $\left\{ I^{(k)}_j \,\middle|\, k = 1, \ldots, n,\ j = 1, \ldots, k-1 \right\}$ in the pencil $\{\cdot,\cdot\}_{\mathsf{E}+\lambda\mathsf{J}^2}$. Use the expansion (2.10) in the identities (2.13) to get $\left\{I^{(k)}_i, G\right\} = \left\{I^{(k-1)}_{i-1}, G\right\}_{\mathsf{J}^2}$ for all $i = 1, \ldots, k-1$ and $k = 1, \ldots, n$. In particular, taking $G = I^{(l)}_j$ for all $j = 0, \ldots, l$ and $l = 1\ldots, n$, we get $\left\{I^{(k)}_i, I^{(l)}_j\right\} = \left\{I^{(k-1)}_{i-1}, I^{(l)}_j\right\}_{\mathsf{J}^2}$, which proves, using Step 1, that the family of functions $\left\{ I^{(k)}_j \,\middle|\, k = 1, \ldots, n,\ j = 1, \ldots, k-1 \right\}$ is also in involution relative to the Poisson bracket $\{\cdot,\cdot\}_{\mathsf{J}^2}$. Therefore, this family of functions is also in involution relative to the pencil of compatible Poisson brackets $\{\cdot,\cdot\}_{\mathsf{E}+\lambda\mathsf{J}^2} = \{\cdot,\cdot\} + \lambda\{\cdot,\cdot\}_{\mathsf{J}^2}$, which proves the theorem. □

### 2.5 Involution in the commutative ring of all Casimir functions of the pencil

Let $\mathcal{F}_{\mathsf{J}}$ be the commutative ring generated by $\left\{ I^{(k)}_j \,\middle|\, k = 1, \ldots, n,\ j = 0, \ldots, k-1 \right\}$ relative to the usual multiplication of functions. By Theorem 2.3, any two elements of $\mathcal{F}_{\mathsf{J}}$ commute in all Poisson brackets $\{\cdot,\cdot\}_{\mathsf{E}+\lambda\mathsf{J}^2}$. Let $\mathcal{G}_{\mathsf{J}}$ be the commutative ring of rational functions in $\lambda$ with coefficients in $\mathcal{F}_{\mathsf{J}}$ with respect to the usual function multiplication. By the Leibniz identity for the Poisson bracket, it follows that all elements of $\mathcal{G}_{\mathsf{J}}$ are in involution in all Poisson brackets $\{\cdot,\cdot\}_{\mathsf{E}+\lambda\mathsf{J}^2}$. Let $\mathcal{F}_{\mathcal{P}}$ be the commutative ring, relative to the usual multiplication of functions, generated by the Casimir functions of all Poisson brackets $\{\cdot,\cdot\}_{\mathsf{E}+\lambda\mathsf{J}^2}$. It is shown in [8, Proposition 1] that the elements of $\mathcal{F}_{\mathcal{P}}$ are in involution in all Poisson brackets $\{\cdot,\cdot\}_{\mathsf{E}+\lambda\mathsf{J}^2}$. For the sake of completeness, we shall give a proof of this result below by establishing a tight relation between the commutative ring $\mathcal{F}_{\mathcal{P}}$ and the family of functions $\left\{ I^{(k)}_j \,\middle|\, k = 1, \ldots, n,\ j = 0, \ldots, k-1 \right\}$.

**Theorem 2.4.** *We have $\mathcal{G}_{\mathsf{J}} = \mathcal{F}_{\mathcal{P}}$ as commutative rings with respect to the usual multiplication of functions. Therefore, the elements of $\mathcal{F}_{\mathcal{P}}$ commute in all Poisson brackets $\{\cdot,\cdot\}_{\mathsf{E}+\lambda\mathsf{J}^2}$.*

*Proof.* We begin with an explicit description of the generators $I^{(k)}_j(X) = \dfrac{1}{k}\mathrm{Tr}\left(C^{(k)}_j(X)\right)$ of $\mathcal{F}_{\mathsf{J}}$. Since

$$\left(\sqrt{-1}X + \lambda\mathsf{J}^2\right)^k = \sum_{j=0}^{k}\left(\sqrt{-1}\right)^{k-j}\left(\sum_{l_1+\cdots+l_k=k-j,\, m_1+\cdots+m_k=j} X^{l_1}\mathsf{J}^{2m_1}\cdots X^{l_k}\mathsf{J}^{2m_k}\right)\lambda^j,$$

where the summation is taken over all $l_\alpha = 0, 1$, $m_\beta = 0, 1$, and $l_\alpha + m_\alpha = 1$, $\alpha, \beta = 1, \ldots, k$, we conclude that

$$C^{(k)}_j(X) = \left(\sqrt{-1}\right)^{k-j}\sum_{l_1+\cdots+l_k=k-j,\, m_1+\cdots+m_k=j} X^{l_1}\mathsf{J}^{2m_1}\cdots X^{l_k}\mathsf{J}^{2m_k}$$

and hence

$$I_j^{(k)}(X) = \frac{1}{k}\left(\sqrt{-1}\right)^{k-j} \operatorname{Tr}\left(\sum_{l_1+\cdots+l_k=k-j,\, m_1+\cdots+m_k=j} X^{l_1}\mathsf{J}^{2m_1}\cdots X^{l_k}\mathsf{J}^{2m_k}\right) \in \mathbb{R}. \tag{2.14}$$

Indeed, each product in the sum under the trace contains exactly $k-j$ matrices $X \in \mathfrak{u}(n)$ and so $\sqrt{-1}X$ is Hermitian, which renders the whole expression real.

Next, we describe the generators of $\mathcal{F}_{\mathcal{P}}$. To do this, we first determine all Casimir functions of $\mathfrak{u}(n)^*_{\mathsf{A}}$, where $\mathsf{A}$ is an $n\times n$ positive-definite Hermitian matrix. Let $\mathsf{D} = \operatorname{diag}(d_1,\ldots,d_n)$, $d_j > 0$, $j = 1,\ldots,n$, be the real diagonal matrix of its eigenvalues; thus there is a unitary matrix $g \in U(n)$ such that $\mathsf{A} = g^*\mathsf{D}g$. The square root of $\mathsf{A}$ is defined by $\sqrt{\mathsf{A}} := g^*\sqrt{\mathsf{D}}g$, where $\sqrt{\mathsf{D}} = \operatorname{diag}\left(\sqrt{d_1},\ldots,\sqrt{d_n}\right)$ and $\sqrt{d_j} > 0$ is chosen to be the positive square root of $d_j$. Define $\Psi_{\mathsf{A}} : \mathfrak{u}(n)_{\mathsf{A}} \to \mathfrak{u}(n)$ by $\Psi_{\mathsf{A}}(X) := \sqrt{\mathsf{A}}X\sqrt{\mathsf{A}}$. This map is clearly linear and invertible since $\Psi_{\mathsf{A}}^{-1} = \Psi_{\mathsf{A}^{-1}}$. In addition,

$$\begin{aligned}
\Psi_{\mathsf{A}}\left([X,Y]_{\mathsf{A}}\right) &= \sqrt{\mathsf{A}}\left(X\mathsf{A}Y - Y\mathsf{A}X\right)\sqrt{\mathsf{A}}\\
&= \left(\sqrt{\mathsf{A}}X\sqrt{\mathsf{A}}\right)\left(\sqrt{\mathsf{A}}Y\sqrt{\mathsf{A}}\right) - \left(\sqrt{\mathsf{A}}Y\sqrt{\mathsf{A}}\right)\left(\sqrt{\mathsf{A}}X\sqrt{\mathsf{A}}\right)\\
&= [\Psi_{\mathsf{A}}(X), \Psi_{\mathsf{A}}(Y)]
\end{aligned}$$

which shows that $\Psi_{\mathsf{A}} : \mathfrak{u}(n)_{\mathsf{A}} \to \mathfrak{u}(n)$ is a Lie algebra isomorphism. Therefore, $\Psi^*_{\mathsf{A}} : \mathfrak{u}(n)^* = \mathfrak{u}(n) \to \mathfrak{u}(n)^*_{\mathsf{A}} = \mathfrak{u}(n)_{\mathsf{A}}$ is an isomorphism of Lie-Poisson spaces and, consequently, the Casimir functions of $\mathfrak{u}(n)_{\mathsf{A}}$ are $C \circ (\Psi^*_{\mathsf{A}})^{-1}$, where $C$ is an arbitrary Casimir function of $\mathfrak{u}(n)$. Since for any $Z \in \mathfrak{u}(n)$ we have

$$\begin{aligned}
\langle \Psi^*_{\mathsf{A}}(X), Z\rangle = \langle X, \Psi_{\mathsf{A}}(Z)\rangle &= \left\langle X, \sqrt{\mathsf{A}}Z\sqrt{\mathsf{A}}\right\rangle = -\operatorname{Tr}\left(X\sqrt{\mathsf{A}}Z\sqrt{\mathsf{A}}\right)\\
&= -\operatorname{Tr}\left(\sqrt{\mathsf{A}}X\sqrt{\mathsf{A}}Z\right) = \langle \Psi_{\mathsf{A}}(X), Z\rangle,
\end{aligned}$$

it follows that $\Psi^*_{\mathsf{A}} = \Psi_{\mathsf{A}}$ and hence $(\Psi^*_{\mathsf{A}})^{-1} = \Psi_{\mathsf{A}^{-1}}$. Therefore, all the Casimir functions of $\mathfrak{u}(n)^*_{\mathsf{A}}$ are all of the form $X \mapsto C\left(\sqrt{\mathsf{A}^{-1}}X\sqrt{\mathsf{A}^{-1}}\right)$ for any Casimir function $C$ of $\mathfrak{u}(n)$. However, all Casimir functions of $\mathfrak{u}(n)$ are arbitrary smooth functions of $\operatorname{Tr}(X^k)$. Since

$$\operatorname{Tr}\left(\left(\sqrt{\mathsf{A}^{-1}}X\sqrt{\mathsf{A}^{-1}}\right)^k\right) = \operatorname{Tr}\left(\left(X\mathsf{A}^{-1}\right)^k\right),$$

we conclude that all Casimir functions of $\mathfrak{u}(n)^*_{\mathsf{A}}$ are generated by the functions $X \mapsto \dfrac{1}{k}\operatorname{Tr}\left(\left(X\mathsf{A}^{-1}\right)^k\right)$, $k = 1,\ldots,n$.

In particular, if $|\lambda|$ is small, then $\mathsf{A} := \mathsf{E} + \lambda\mathsf{J}^2$ is a positive definite $n\times n$ Hermitian matrix and hence the following functions generate the ring of Casimir functions on $\mathfrak{u}(n)^*_{\mathsf{A}}$:

$$\begin{aligned}
\frac{(-1)^r}{r}\operatorname{Tr}\left(\left(X\left(\mathsf{E}+\lambda\mathsf{J}^2\right)^{-1}\right)^r\right) &= \frac{(-1)^r}{r}\operatorname{Tr}\left(\left(X\sum_{l=0}^{\infty}\left(-\lambda\mathsf{J}^2\right)^l\right)^r\right)\\
&= \frac{(-1)^r}{r}\operatorname{Tr}\left(\sum_{m=0}^{\infty}(-1)^m\left(\sum_{j_1+\cdots+j_r=m} X\mathsf{J}^{2j_1}\cdots X\mathsf{J}^{2j_r}\right)\lambda^m\right)\\
&= \sum_{m=0}^{\infty}(-1)^{r+m}\frac{1}{r}\operatorname{Tr}\left(\sum_{j_1+\cdots+j_r=m} X\mathsf{J}^{2j_1}\cdots X\mathsf{J}^{2j_r}\right)\lambda^m.
\end{aligned} \tag{2.15}$$

In the summation, the exponents $j_1, \ldots, j_r$ are integers in $\{0, 1, \ldots, m\}$. This series expansion is convergent for $|\lambda|$ small and all its coefficients are in $\mathcal{F}_{\mathsf{J}}$. The trace in the last expression contains the sum of all possible products with $r$ matrices $X$ and $m$ matrices $\mathsf{J}^2$, without redundancy. This shows that $\mathcal{F}_{\mathcal{P}} \subseteq \mathcal{G}_{\mathsf{J}}$.

Conversely, from (2.14), since $l_\alpha = 0, 1$, $m_\alpha = 0, 1$, $l_\alpha + m_\alpha = 1$, $m_1 + \cdots + m_k = j$, and $l_1 + \cdots + l_k = k - j$, we conclude that under the trace there is the sum over all possible products of $k - j$ matrices $X$ and $j$ matrices $\mathsf{J}^2$ without redundancy. But this is precisely the structure of the matrix under the trace in the coefficient of $\lambda^j$ in (2.15) for $r = k - j$: the sum of all possible products containing $k - j$ matrices $X$ and $j$ matrices $\mathsf{J}^2$ without redundancy. Hence $\mathcal{G}_{\mathsf{J}} \subseteq \mathcal{F}_{\mathcal{P}}$.

We conclude that $\mathcal{G}_{\mathsf{J}} = \mathcal{F}_{\mathcal{P}}$. □

In the proof of Theorem 2.3, we have used the Lie algebra isomorphism $\Psi_{\mathsf{A}} : \mathfrak{u}(n)_{\mathsf{A}} \to \mathfrak{u}(n)$ defined for each positive-definite Hermitian matrix $\mathsf{A}$. This Lie algebra isomorphism can be generalized to an arbitrary non-degenerate Hermitian $n \times n$ matrix $\mathsf{A}$. Indeed, there is a unitary matrix $g \in U(n)$ such that $\mathsf{A} = g^* \mathsf{D} g$, where $\mathsf{D} = \operatorname{diag}(d_1, \ldots, d_n)$ is the real diagonal matrix of eigenvalues of $\mathsf{A}$. For simplicity, assume that $d_1, \ldots, d_p > 0$ and $d_{p+1}, \ldots, d_n < 0$. Let $\sqrt{\mathsf{D}} := \operatorname{diag}(\sqrt{d_1}, \ldots, \sqrt{d_n})$, where positive square roots are chosen, i.e., $\sqrt{d_j} > 0$ if $d_j > 0$ and $\operatorname{Im}\sqrt{d_j} > 0$ if $d_j < 0$. Define the square root of $\mathsf{A}$ by $\sqrt{\mathsf{A}} = g^* \sqrt{D} g$.

**Proposition 2.5.** *Let $\mathsf{A}$ be a non-degenerate Hermitian matrix having $p$ positive and $q := n - p$ negative eigenvalues. The mapping $\Psi_{\mathsf{A}} : \mathfrak{u}(n)_{\mathsf{A}} \ni X \mapsto \sqrt{\mathsf{A}} X \sqrt{\mathsf{A}} \in \mathfrak{u}(p, q)$ is a Lie algebra isomorphism. Here, $\mathfrak{u}(p, q) := \{X \in \mathbb{C}^{n\times n} \mid X^* \mathsf{E}_{p,q} + \mathsf{E}_{p,q} X = 0\}$, where $\mathsf{E}_{p,q} = \operatorname{diag}(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q})$.*

The proof of this proposition is formally the same as in the case where $\mathsf{A} > 0$, which is described in the proof of Theorem 2.3.

## 2.6 The relationship between the $\mathfrak{u}(n)$- and $\mathfrak{su}(n)$-Euler equations

The generalized inertia tensor $\mathcal{J}$ for the $U(n)$ free rigid body is a sectional operator, as defined in [14, Chapter 2, §6]. To see this, let $\mathfrak{u}(n) = \mathfrak{h}_0 \dot{+} \mathfrak{m}_0$ (direct sum of vector spaces) be the Cartan decomposition of $\mathfrak{u}(n)$, where $\mathfrak{h}_0$ is the commutative Lie algebra consisting of diagonal matrices with purely imaginary entries and $\mathfrak{m}_0$ is the orthogonal complement of $\mathfrak{h}_0$ with respect to $\langle \cdot, \cdot \rangle$, i.e., the vector space consisting of all the skew-Hermitian matrices with zero diagonal.

As we have seen in Lemma 2.1, we can assume, without loss of generality, that the Hermitian matrix $\mathsf{J}$ is real and diagonal. Then, $\sqrt{-1}\mathsf{J}$, $\sqrt{-1}\mathsf{J}^2 \in \mathfrak{h}_0$ and a direct computation shows that the inertia tensor can be written as

$$\mathcal{J}(X) = \left(\operatorname{ad}_{\sqrt{-1}\mathsf{J}}\right)^{-1}\left(\operatorname{ad}_{\sqrt{-1}\mathsf{J}^2}(X)\right) + D\left(\operatorname{pr}(X)\right), \quad X = (x_{ij}) \in \mathfrak{u}(n), \tag{2.16}$$

where the projection $\operatorname{pr} : \mathfrak{u}(n) \to \mathfrak{h}_0$ and the linear isomorphism $D : \mathfrak{h}_0 \to \mathfrak{h}_0$ are given, respectively, by

$$\operatorname{pr}\left((x_{ij})\right) := \operatorname{diag}\left(x_{11}, \ldots, x_{nn}\right), \qquad D\left(\operatorname{diag}\left(x_1, \ldots, x_n\right)\right) := \operatorname{diag}\left(\frac{x_1}{2J_1}, \ldots, \frac{x_n}{2J_n}\right). \tag{2.17}$$

Note that we have $\operatorname{ad}_{\sqrt{-1}\mathsf{J}^2}(X) \in \mathfrak{m}_0$ for any $X \in \mathfrak{u}(n)$ and that the linear mapping $\operatorname{ad}_{\mathsf{H}}|_{\mathfrak{m}_0} : \mathfrak{m}_0 \to \mathfrak{m}_0$ is invertible for any invertible diagonal matrix $\mathsf{H} \in \mathfrak{h}_0$. Since the operator $\mathcal{J}$ leaves both $\mathfrak{h}_0$ and $\mathfrak{m}_0$ invariant, it follows that $\mathcal{J}$ *is a sectional operator* in the sense of [14, Chapter 2, §6] (see also Remark 2.3 below).

**Remark 2.3.** In [26, 27], all systems described by the Euler equation

$$\frac{\mathsf{d}}{\mathsf{d}t}X = [X, \varphi_{a,b,D}(X)] \tag{2.18}$$

on an arbitrary complex semi-simple Lie algebra $\mathfrak{g}$ are studied. Here, $X \in \mathfrak{g}$ and the linear operator $\varphi_{a,b,D} : \mathfrak{g} \to \mathfrak{g}$ is defined by $\varphi_{a,b,D}(X) = \left((\mathrm{ad}_a)^{-1} \circ \mathrm{ad}_b\right)(X') + D(T)$, where $a, b \in \mathfrak{h}$ are two generic (i.e., regular semi-simple) elements in some Cartan subalgebra $\mathfrak{h}$, $X = X' + T$ with $T \in \mathfrak{h}$ and $X' \in \mathfrak{m}$, the sum of all root spaces in the root space decomposition induced by $\mathfrak{h}$, and $D : \mathfrak{h} \to \mathfrak{h}$ is a symmetric invertible linear operator with respect to the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$. The *sectional operator* $\varphi_{a,b,D}$ is a symmetric operator with respect to the Killing form; see [26, §4] and [27, Section 2]. The remarkable main result in [26, 27] is the complete integrability of the Euler equation (2.18) on $\mathfrak{g}$ (and hence clearly of its restriction to its normal (split) real form $\mathfrak{g}_{\mathbb{R}}$) as well as of its restrictions to the compact real form $\mathfrak{g}_u$ and the normal-compact real form $\mathfrak{g}_n := \mathfrak{g}_u \cap \mathfrak{g}_{\mathbb{R}}$.

Although the Lie algebra $\mathfrak{u}(n)$ is not semi-simple, the description of the Euler equation (2.18) can be extended to $\mathfrak{u}(n)$, as we have seen in (2.16). In fact, Mishchenko and Fomenko mentioned the case $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{g}_u = \mathfrak{u}(n)$, and $\mathfrak{g}_n = \mathfrak{so}(n)$, assuming $D = 0$. In particular, the restriction to $\mathfrak{so}(n)$ is nothing but the $SO(n)$ free rigid body. However, the restriction of the system on $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ to $\mathfrak{g}_u = \mathfrak{u}(n)$ is not discussed in detail in [26, 27], since the main interest of these papers is the case of semi-simple Lie algebras. Note that in order to obtain the $SO(n)$ free rigid body, it is not necessary to assume that $D = 0$ in the $\mathfrak{gl}(n, \mathbb{C})$ free rigid body, since any choice of $D$ leads to the same $SO(n)$ free rigid body. ♢

Returning to the Euler equation (2.4) for the $U(n)$ free rigid body dynamics, we can restrict it to any level hyperplane $\left\{X \in \mathfrak{u}(n) \,\middle|\, I_0^{(1)}(X) = \sqrt{-1}\mathrm{Tr}(X) = c\right\}$, where $c \in \mathbb{R}$ is an arbitrary constant, since $I_0^{(1)}$ is a Casimir function with respect to the Poisson bracket $\{\cdot, \cdot\}$. Concerning the Mishchenko-Fomenko free rigid body, we have the following result.

**Proposition 2.6.** *The restriction of the Euler equation* (2.4) *for the* $U(n)$ *free rigid body to* $\mathfrak{su}(n)$ *admits the Mishchenko-Fomenko formulation*

$$\frac{\mathsf{d}X}{\mathsf{d}t} = [X, \varphi_{a',b',D'}(X)],$$

*where* $X \in \mathfrak{su}(n)$,

$$\begin{aligned}
a' &= \sqrt{-1}\mathrm{diag}\left(J_1^2 - \frac{1}{n}\sum_{i=1}^n J_i^2, \cdots, J_n^2 - \frac{1}{n}\sum_{i=1}^n J_i^2\right),\\
b' &= \sqrt{-1}\mathrm{diag}\left(J_1 - \frac{1}{n}\sum_{i=1}^n J_i, \cdots, J_n - \frac{1}{n}\sum_{i=1}^n J_i\right),\\
D'&\left(\sqrt{-1}\mathrm{diag}\left(x_1, x_2 - x_1, \cdots, x_{n-1} - x_{n-2}, -x_{n-1}\right)\right)\\
&\qquad = \sqrt{-1}\mathrm{diag}\left(\frac{x_i - x_{i-1}}{2J_i} - \frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{1}{2J_k} - \frac{1}{2J_{k+1}}\right)x_k\right)_{i=1,\ldots,n},
\end{aligned}$$

*and we define* $x_0 := x_n := 0$.

*Proof.* The Euler equation (2.4) can be written as

$$\frac{\mathsf{d}X}{\mathsf{d}t} = \left[X, \mathcal{J}^{-1}(X) - \frac{1}{n}\mathrm{Tr}\left(\mathcal{J}^{-1}(X)\right)\mathsf{E}\right]. \tag{2.19}$$

For $1 \leq i \neq j \leq n$, the $(i,j)$ component of the matrix $\mathcal{J}^{-1}(X) - \frac{1}{n}\mathrm{Tr}\left(\mathcal{J}^{-1}(X)\right)\mathsf{E} \in \mathfrak{su}(n)$ is

$$\frac{x_{ij}}{J_i + J_j},$$

where $X = (x_{ij}) \in \mathfrak{su}(n)$, while the diagonal components are

$$\frac{x_{ii}}{2J_i} - \frac{1}{n}\sum_{k=1}^{n}\frac{x_{kk}}{2J_k}, \qquad i = 1, \ldots, n, \tag{2.20}$$

where $x_{ii} = \sqrt{-1}\,(x_i - x_{i-1})$, $x_0 = x_n = 0$. Note that the expression (2.20) is rewritten as

$$\begin{aligned}
&\sqrt{-1}\left(\frac{x_i - x_{i-1}}{2J_i} - \frac{1}{n}\sum_{k=1}^{n}\frac{x_k - x_{k-1}}{2J_k}\right) \\
&\qquad = \sqrt{-1}\left(\frac{x_i - x_{i-1}}{2J_i} - \frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{1}{2J_k} - \frac{1}{2J_{k+1}}\right)x_k\right), \qquad i = 1, \ldots, n.
\end{aligned}$$

Thus, we see that the operator $D'$ on the Cartan subalgebra $\mathfrak{h}_0 \cap \mathfrak{su}(n)$ of $\mathfrak{su}(n)$ coincides with the restriction of the operator

$$\widetilde{D} : \mathfrak{su}(n) \ni X \longmapsto \mathcal{J}^{-1}(X) - \frac{1}{n}\mathrm{Tr}\left(\mathcal{J}^{-1}(X)\right)\mathsf{E} \in \mathfrak{su}(n)$$

to $\mathfrak{h}_0 \cap \mathfrak{su}(n)$. Recall that $\mathfrak{u}(n) = \mathfrak{h}_0 \dot{+} \mathfrak{m}_0$ is the Cartan decomposition, where $\mathfrak{h}_0$ is the commutative Lie algebra consisting of diagonal matrices with purely imaginary entries and $\mathfrak{m}_0$, the $\langle\cdot,\cdot\rangle$-orthogonal complement of $\mathfrak{h}_0$, is the real vector space of all skew-Hermitian matrices with zero diagonal.

Moreover, the operator $\widetilde{D} : \mathfrak{su}(n) \to \mathfrak{su}(n)$ is symmetric with respect to the inner product $\langle\cdot,\cdot\rangle$. Indeed, for any $X, Y \in \mathfrak{su}(n)$, we have

$$\begin{aligned}
\left\langle \widetilde{D}(X), Y \right\rangle &= \left\langle \mathcal{J}^{-1}(X) - \frac{1}{n}\mathrm{Tr}\left(\mathcal{J}^{-1}(X)\right)\mathsf{E}, Y \right\rangle \\
&= \left\langle \mathcal{J}^{-1}(X), Y \right\rangle - \frac{1}{n}\mathrm{Tr}\left(\mathcal{J}^{-1}(X)\right)\langle \mathsf{E}, Y\rangle \\
&= \left\langle X, \mathcal{J}^{-1}(Y) \right\rangle \\
&= \left\langle X, \widetilde{D}(Y) \right\rangle,
\end{aligned}$$

since $\langle \mathsf{E}, Y\rangle = -\mathrm{Tr}(Y) = 0$. Thus, the restriction $D'$ of $\widetilde{D}$ to $\mathfrak{h}_0 \cap \mathfrak{su}(n)$ is also symmetric with respect to $\langle\cdot,\cdot\rangle|_{\mathfrak{h}_0 \cap \mathfrak{su}(n)}$.

Next, define $a := \sqrt{-1}\mathrm{diag}(J_1^2, \ldots, J_n^2)$, $b := \sqrt{-1}\mathrm{diag}(J_1, \ldots, J_n)$ and note that $a' = a - \frac{1}{n}(\mathrm{Tr}(a))\mathsf{E}$, $b' = b - \frac{1}{n}(\mathrm{Tr}(b))\mathsf{E}$. Then we have

$$\left(\mathrm{ad}_{a'}^{-1} \circ \mathrm{ad}_{b'}\right)(X) = \left(\mathrm{ad}_{a}^{-1} \circ \mathrm{ad}_{b}\right)(X), \qquad X \in \mathfrak{su}(n).$$

From (2.16), it follows that the restriction of the operator $\mathcal{J}^{-1}$ to $\mathfrak{m}_0 = \mathfrak{m}_0 \cap \mathfrak{su}(n)$ coincides with $\mathrm{ad}_{a'}^{-1} \circ \mathrm{ad}_{b'}$.

Finally, the arguments given above show that

$$\varphi_{a',b',D'}(X) = \varphi_{a,b,\widetilde{D}}(X) = \mathcal{J}^{-1}(X) - \frac{1}{n}\mathrm{Tr}(\mathcal{J}^{-1}(X))\mathsf{E}$$

and $[X, \varphi_{a',b',D'}(X)] = [X, \varphi_{a,b,\widetilde{D}}(X)] = [X, \mathcal{J}^{-1}(X)]$, for all $X \in \mathfrak{su}(n)$, which proves the proposition. $\square$

As a Lie algebra, $\mathfrak{u}(n)$ is decomposed into the direct sum $\mathfrak{su}(n) \oplus \mathfrak{z}$, where the center $\mathfrak{z}$ consists of purely imaginary multiples of the identity. The previous proposition shows that the $U(n)$ free rigid body dynamics naturally restricts to the Mishchenko-Fomenko free rigid body system on $\mathfrak{su}(n)$. Since $I_0^{(1)}(X) = \sqrt{-1}\mathrm{Tr}(X)$ is a Casimir function of the Lie-Poisson bracket $\{\cdot,\cdot\}$ on $\mathfrak{u}(n)$, the $U(n)$ free rigid body Euler equation leaves the level hyperplanes of $I_0^{(1)}$ invariant; these are Poisson submanifolds of $(\mathfrak{u}(n), \{\cdot,\cdot\})$. For any $c \in \mathbb{R}$, the level hyperplane $\left(I_0^{(1)}\right)^{-1}(c)$ is mapped to $\mathfrak{su}(n) = \left(I_0^{(1)}\right)^{-1}(0)$ by the translation

$$\phi_c : \left(I_0^{(1)}\right)^{-1}(c) \ni X \mapsto X + \frac{\sqrt{-1}c}{n}\mathsf{E} \in \mathfrak{su}(n).$$

Thus, for any $f, g \in \mathcal{C}^\infty(\mathfrak{u}(n))$ and $X \in \left(I_0^{(1)}\right)^{-1}(c) \subset \mathfrak{u}(n)$, we have

$$\begin{aligned}
\{f \circ \phi_c, g \circ \phi_c\}(X) &= \langle X, [\nabla(f \circ \phi_c)(X), \nabla(g \circ \phi_c)(X)]\rangle \\
&= \langle X, [\nabla f(\phi_c(X)), \nabla g(\phi_c(X))]\rangle \\
&= \left\langle X + \frac{\sqrt{-1}c}{n}\mathsf{E}, [\nabla f(\phi_c(X)), \nabla g(\phi_c(X))]\right\rangle \\
&= \langle \phi_c(X), [\nabla f(\phi_c(X)), \nabla g(\phi_c(X))]\rangle \\
&= \{f, g\}(\phi_c(X)),
\end{aligned}$$

which shows that $\phi_c : \left(I_0^{(1)}\right)^{-1}(c) \to \mathfrak{su}(n)$ is a Poisson isomorphism. Note that for any $Y \in \mathfrak{u}(n)$ we have $\langle \nabla(f \circ \phi_c)(X), Y\rangle = \mathsf{d}(f \circ \phi_c)(X) \cdot Y = \mathsf{d}f(\phi_c(X)) \cdot (T\phi_c(X))Y = \mathsf{d}f(\phi_c(X)) \cdot Y = \langle \mathsf{d}f(\phi_c(X)), Y\rangle$, and hence $\nabla(f \circ \phi_c)(X) = \nabla f(\phi_c(X))$. Consequently, $\phi_c$ maps the $U(n)$ free rigid body Euler equation $\frac{\mathsf{d}}{\mathsf{d}t}X = \left[X, \mathcal{J}^{-1}(X)\right]$, for $X \in \left(I_0^{(1)}\right)^{-1}(c)$, to the Lie-Poisson equation

$$\frac{\mathsf{d}X}{\mathsf{d}t} = \left[X, \mathcal{J}^{-1}(X) - \frac{\sqrt{-1}c}{n}\mathcal{J}^{-1}(\mathsf{E})\right], \qquad X \in \mathfrak{su}(n), \tag{2.21}$$

on $\mathfrak{su}(n)$. The Hamiltonian function of this system is

$$\frac{1}{2}\left\langle X, \mathcal{J}^{-1}(X)\right\rangle - \frac{\sqrt{-1}c}{n}\left\langle \mathcal{J}^{-1}(\mathsf{E}), X\right\rangle, \qquad X \in \mathfrak{su}(n).$$

We show that the function

$$L(X) := \frac{\sqrt{-1}c}{n}\left\langle \mathcal{J}^{-1}(\mathsf{E}), X\right\rangle, \qquad X \in \mathfrak{su}(n),$$

Poisson commutes with any function in $\mathcal{F}_{\mathsf{J}}$. Indeed,

$$\begin{aligned}
\{L, f_k\}(X) &= \langle X, [\nabla L(X), \nabla f_k(X)]\rangle \\
&= \left\langle X, \left[\frac{\sqrt{-1}c}{n}\mathcal{J}^{-1}(\mathsf{E}), \sqrt{-1}\left(\sqrt{-1}X + \lambda\mathsf{E}\right)^{k-1}\right]\right\rangle \\
&= \frac{\sqrt{-1}c}{n}\left\langle \sqrt{-1}X, \left[\mathcal{J}^{-1}(\mathsf{E}), \left(\sqrt{-1}X + \lambda\mathsf{E}\right)^{k-1}\right]\right\rangle \\
&= \frac{\sqrt{-1}c}{n}\left\langle \left[\left(\sqrt{-1}X + \lambda\mathsf{E}\right)^{k-1}, \sqrt{-1}X\right], \mathcal{J}^{-1}(\mathsf{E})\right\rangle \\
&= \frac{\sqrt{-1}c}{n}\left\langle \left[\left(\sqrt{-1}X + \lambda\mathsf{E}\right)^{k-1}, -\lambda\mathsf{E}\right], \mathcal{J}^{-1}(\mathsf{E})\right\rangle \\
&= -\frac{\sqrt{-1}c}{n}\left\langle \left(\sqrt{-1}X + \lambda\mathsf{E}\right)^{k-1}, \left[\lambda\mathsf{E}, \mathcal{J}^{-1}(\mathsf{E})\right]\right\rangle \\
&= 0.
\end{aligned}$$

In addition, we show now that the function $L$ can be written as a linear combination of the restrictions of the functions in $\mathcal{F}_{\mathsf{J}}$ to $\mathfrak{su}(n)$. To see this, we take the functions

$$I^{(k)}_{k-1}(X) = \operatorname{Tr}\left(\mathsf{J}^{2k-2}X\right) = \sum_{l=1}^{n} J_l^{2k-2}x_{ll}, \quad k = 1, \ldots, n,$$

and consider their linear combination

$$L'(X) := \sum_{k=1}^{n}\sum_{j=1}^{n} J_j^{-1}b_{kj}I^{(k)}_{k-1}(X), \quad X \in \mathfrak{su}(n),$$

where the matrix $B := (b_{ij})$ is the inverse of the Vandermonde matrix:

$$B^{-1} = \begin{bmatrix} 1 & J_1^2 & \cdots & J_1^{2n-2} \\ \vdots & \vdots & & \vdots \\ 1 & J_n^2 & \cdots & J_n^{2n-2} \end{bmatrix},$$

which is invertible if $J_1^2, \ldots, J_n^2$ are distinct. Since $\sum_{j=1}^{n} J_i^{2j-2}b_{jk} = \delta_{ik}$, we have

$$\begin{aligned}
L'(X) &= \sum_{k=1}^{n}\sum_{j=1}^{n} J_j^{-1}b_{kj}\sum_{l=1}^{n} J_l^{2k-2}x_{ll} \\
&= \sum_{j=1}^{n}\sum_{l=1}^{n} J_j^{-1}\sum_{k=1}^{n} b_{kj}J_l^{2k-2}x_{ll} \\
&= \sum_{j=1}^{n}\sum_{l=1}^{n} J_j^{-1}\delta_{lj}x_{ll} \\
&= \sum_{j=1}^{n} J_j^{-1}x_{jj} = L(X).
\end{aligned}$$

This shows $L \in \mathcal{F}_{\mathsf{J}}|_{\mathfrak{su}(n)}$. To sum up, we have the following theorem.

**Theorem 2.7.** *The restriction of the Euler equation* (2.4) *for the* $U(n)$ *free rigid body to each level hyperplane* $\left\{X \in \mathfrak{u}(n) \,\middle|\, I_0^{(1)} = \sqrt{-1}\operatorname{Tr}(X) = c\right\}$*, where* $c \in \mathbb{R}$ *is an arbitrary constant, can be described by Hamilton's equation on* $(\mathfrak{su}(n), \{\cdot,\cdot\})$ *with respect to the Hamiltonian* $H - L \in \mathcal{F}_{\mathsf{J}}|_{\mathfrak{su}(n)}$ *(the ring of all functions in* $\mathcal{F}_{\mathsf{J}}$ *restricted to* $\mathfrak{su}(n)$*), where*

$$H(X) = \frac{1}{2}\left\langle X, \mathcal{J}^{-1}(X)\right\rangle, \qquad L(X) = \frac{\sqrt{-1}c}{n}\left\langle \mathcal{J}^{-1}(\mathsf{E}), X\right\rangle, \qquad X \in \mathfrak{su}(n).$$

For real constants $c \neq 0$, the restriction of the Euler equation (2.4) for the $U(n)$ free rigid body dynamics is not the Euler equation for the Mishchenko-Fomenko $SU(n)$ free rigid body dynamics whose Hamiltonian is homogeneous quadratic, since the Hamiltonian of the equation (2.21) has the nontrivial linear term $L(X)$. Nevertheless, the Hamiltonian $H - L$ of (2.21) is included in the commutative ring $\mathcal{F}_{\mathsf{J}}|_{\mathfrak{su}(n)}$. Note that the ring $\mathcal{F}_{\mathsf{J}}|_{\mathfrak{su}(n)}$ includes all the Manakov integrals for the Mishchenko-Fomenko free rigid body dynamics on $\mathfrak{su}(n)$ [26, 27]. Thus, the complete integrability of the system (2.21) follows, by using the results of Mishchenko-Fomenko [26, 27]. In the next section, however, we give another proof of the complete integrability, using the method introduced by Bolsinov and Oshemkov [8], since this proof does not require considerations of the restriction of the system, but can be performed directly.

# 3 Complete integrability

In this section, we prove the complete integrability of the $U(n)$ free rigid body, by showing the completeness of the set $\mathcal{F}_{\mathcal{P}}$, or equivalently $\mathcal{G}_{\mathsf{J}}$, of first integrals. In [8], Bolsinov and Oshemkov give a criterion, called by them "the codimension two principle", which implies completeness. Although completeness can be shown as a direct consequence of Theorem 2.7, we prefer a proof of the complete integrability for the $U(n)$ free rigid body dynamics based on the Bolsinov-Oshemkov codimension two principle, first, because it is natural from the viewpoint of the bi-Hamiltonian structure of the $U(n)$ free rigid body dynamics and, second, since the proof can be performed directly, without restricting the $U(n)$ free rigid body dynamics to the level hyperplanes of $I_0^{(1)}$ and then invoking the complete integrability of the $SU(n)$ free rigid body (see, [26, 27, 14]). We emphasize that our proof of the complete integrability gives an application of the Bolsinov-Oshemkov method to a bi-Hamiltonian system on a non semi-simple Lie algebra, a case that is not discussed in detail in [8, 7]. At the end of the section, we also mention another proof of the complete integrability of the $U(n)$ free rigid body that uses a theorem of Brailov on completely involutive sets of functions on affine Lie algebras; see [14, Chapter 5, §20.2] for a nice presentation of this result.

## 3.1 Completeness

**Definition 3.1.** *Let* $(M, \{\,,\})$ *be a Poisson manifold and* $\mathcal{A} : \Omega^1(M) \times \Omega^1(M) \to \mathcal{C}^\infty(M)$ *the associated Poisson tensor. A subspace* $V \subset T_x^*M$ *is called* isotropic*,* $\mathcal{A}(x)(\alpha_x, \beta_x) = 0$ *for all* $\alpha_x, \beta_x \in V$*.*

Define the $\mathcal{A}(x)$-orthogonal complement of $V$ in the usual way, namely

$$V^{\perp_{\mathcal{A}(x)}} = \left\{\beta_x \in T_x^*M \mid \mathcal{A}(x)\left(\alpha_x, \beta_x\right) = 0, \text{ for all } \alpha_x \in V\right\}.$$

Then, it is easy to see that $V$ *is isotropic if and only if* $V \subset V^{\perp_{\mathcal{A}(x)}}$, which is the standard definition for isotropic subspaces (e.g., [32, Chapitre IV, §1.3 Définition 3], [29, 10.4.18]) and agrees with the Bolsinov-Oshemkov approach [8, Section 2], although it is not explicitly written there.

The *rank* of $\mathcal{A}$ at $x \in M$ is, by definition, equal to $\dim \operatorname{range}\left(T_x^*M \ni \alpha_x \mapsto \mathcal{A}(x)(\alpha_x, \cdot) \in T_xM\right)$.

**Lemma 3.1.** *Let $E$ be a finite dimensional real vector space and $\Lambda : E \times E \to \mathbb{R}$ a skew-symmetric bilinear map. If $F \subset E$ is a vector subspace, define $F^{\perp_\Lambda} := \{e \in E \mid \Lambda(e, f) = 0$ for all $f \in F\}$ and denote $\ker \Lambda := E^{\perp_\Lambda}$.*

(i) *Then $F^{\perp_\Lambda}$ is a vector subspace of $E$, $F \subseteq \left(F^{\perp_\Lambda}\right)^{\perp_\Lambda}$, $\ker \Lambda \subseteq F^{\perp_\Lambda}$, and*

$$\dim F + \dim F^{\perp_\Lambda} - \dim\left(F \cap \ker \Lambda\right) = \dim E, \tag{3.1}$$

$$\dim\left(F^{\perp_\Lambda}\right)^{\perp_\Lambda} = \dim(F + \ker \Lambda). \tag{3.2}$$

*Thus, $\ker \Lambda \subseteq F$ if and only if $F = \left(F^{\perp_\Lambda}\right)^{\perp_\Lambda}$. In addition, $\ker \Lambda$ is an isotropic subspace and $\operatorname{rank} \Lambda$ is even.*

(ii) *Let $F_1, F_2 \subset E$ be two vector subspaces. Then $F_1 \subseteq F_2$ implies $F_2^{\perp_\Lambda} \subseteq F_1^{\perp_\Lambda}$. When $F_1$ and $F_2$ are arbitrary, we have $(F_1 + F_2)^{\perp_\Lambda} = F_1^{\perp_\Lambda} \cap F_2^{\perp_\Lambda}$.*

(iii) *Suppose that $F$ is maximal isotropic, i.e., it is a maximal vector subspace relative to inclusion satisfying $F \subseteq F^{\perp_\Lambda}$. Then, $\ker \Lambda \subseteq F$, $F = F^{\perp_\Lambda}$, and*

$$\dim F = \dim E - \frac{1}{2} \operatorname{rank} \Lambda. \tag{3.3}$$

*Conversely, if $F$ is an isotropic subspace whose dimension is given by (3.3), then $F$ is maximal isotropic in $E$. If $\Lambda \neq 0$, the subspace $\ker \Lambda$ is never a maximal isotropic subspace.*

(iv) *Let $F \subset E$ be an isotropic subspace. Assume that $\Lambda \neq 0$ and set $r := \frac{1}{2}\operatorname{rank}\Lambda$. Then, $\dim F = \dim E - r$ if and only if*

  (1) *there exist $r$ linearly independent vectors $f_1, \ldots, f_r \in F \setminus \ker\Lambda$ such that $\Lambda(f_1, \cdot),\ldots,$ $\Lambda(f_r, \cdot) \in E^*$ are also linearly independent and*

  (2) $F = \operatorname{span}\{f_1, \cdots, f_r\} \oplus \ker \Lambda$.

*Proof.* (i) The first three statements are obvious. The dimension formula (3.1) is obtained in the following way. The skew-symmetric bilinear map $\Lambda$ induces a non-degenerate skew-symmetric bilinear map $[\Lambda] : (E/\ker \Lambda) \times (E/\ker \Lambda) \to \mathbb{R}$ by $[\Lambda]([e], [e']) := \Lambda(e, e')$, where $[e] = e + \ker \Lambda \in E/\ker \Lambda$ denotes the equivalence class of $e \in E$. Since $[\Lambda]$ is non-degenerate, it follows that $\dim(E/\ker \Lambda) = \operatorname{rank} \Lambda$ is an even number (see, e.g., [1, Proposition 3.1.3]).

The vector subspace $\{[f] \in E/\ker \Lambda \mid f \in F\}$ is isomorphic to $F/(F \cap \ker \Lambda)$ and

$$\{[f] \in E/\ker \Lambda \mid f \in F\}^{\perp_{[\Lambda]}} = F^{\perp_\Lambda}/\ker \Lambda. \tag{3.4}$$

Since $[\Lambda]$ is non-degenerate, we have (see, e.g., [1, Proposition 5.3.2])

$$\dim\{[f] \in E/\ker \Lambda \mid f \in F\} + \dim\{[f] \in E/\ker \Lambda \mid f \in F\}^{\perp_{[\Lambda]}} = \dim(E/\ker \Lambda)$$

which is equivalent to

$$\dim F - \dim(F \cap \ker \Lambda) + \dim F^{\perp_\Lambda} - \dim \ker \Lambda = \dim E - \dim \ker \Lambda,$$

i.e., to (3.1).

To obtain (3.2), write (3.1) for $F^{\perp_\Lambda}$ and use $\ker \Lambda \subset F^{\perp_\Lambda}$ to get

$$\dim F^{\perp_\Lambda} + \dim\left(F^{\perp_\Lambda}\right)^{\perp_\Lambda} - \dim \ker \Lambda = \dim E = \dim F + \dim F^{\perp_\Lambda} - \dim\left(F \cap \ker \Lambda\right)$$

and hence

$$\dim \left(F^{\perp_\Lambda}\right)^{\perp_\Lambda} = \dim \ker \Lambda + \dim F - \dim \left(F \cap \ker \Lambda\right) = \dim(F + \ker \Lambda),$$

which proves (3.2).

Finally, $\ker \Lambda \subset E = \left(E^{\perp_\Lambda}\right)^{\perp_\Lambda} = (\ker \Lambda)^{\perp_\Lambda}$ which shows that $\ker \Lambda$ is isotropic.

(ii) The first relation is an easy verification. To prove the second, use the first and the inclusions $F_1, F_2 \subset F_1 + F_2$ to conclude $(F_1 + F_2)^{\perp_\Lambda} \subseteq F_1^{\perp_\Lambda} \cap F_2^{\perp_\Lambda}$. Conversely, if $f \in F_1^{\perp_\Lambda} \cap F_2^{\perp_\Lambda}$, then $\Lambda(f, f_1) = \Lambda(f, f_2) = 0$ for all $f_1 \in F_1$ and $f_2 \in F_2$. Therefore $\Lambda(f, f_1 + f_2) = 0$ for all $f_1 + f_2 \in F_1 + F_2$, i.e., $f \in (F_1 + F_2)^{\perp_\Lambda}$ proving $F_1^{\perp_\Lambda} \cap F_2^{\perp_\Lambda} \subseteq (F_1 + F_2)^{\perp_\Lambda}$.

(iii) We have $(F + \ker \Lambda)^{\perp_\Lambda} = F^{\perp_\Lambda} \cap \left(E^{\perp_\Lambda}\right)^{\perp_\Lambda} = F^{\perp_\Lambda} \cap E = F^{\perp_\Lambda} \supseteq F$, because $F$ is isotropic. By (i), we have $(F + \ker \Lambda)^{\perp_\Lambda} \supseteq \ker \Lambda$ and hence $(F + \ker \Lambda)^{\perp_\Lambda} \supseteq F + \ker \Lambda$, i.e., $F + \ker \Lambda$ is also an isotropic subspace. Since $F$ is maximal isotropic and $F \subseteq F + \ker \Lambda$, we must have $F = F + \ker \Lambda$ and hence $\ker \Lambda \subseteq F$. Thus, by (3.1), for a maximal isotropic subspace $F$ in $(E, \Lambda)$ we get the dimension formula

$$\dim F + \dim F^{\perp_\Lambda} = \dim E + \dim \ker \Lambda = 2 \dim E - \operatorname{rank} \Lambda. \tag{3.5}$$

By (3.4) it follows that the vector subspace $V \subseteq E$ is isotropic in $(E, \Lambda)$ if and only if $\{[v] \in E/\ker \Lambda \mid v \in V\}$ is isotropic in $(E/\ker \Lambda, [\Lambda])$. Since any subspace of $E/\ker \Lambda$ is of the form $V/\ker \Lambda$, where $V \subseteq E$ is a vector subspace containing $\ker \Lambda$, the previous statement implies that $F$ is maximal isotropic in $(E, \Lambda)$ if and only if $F/\ker \Lambda$ is maximal isotropic in $(E/\ker \Lambda, [\Lambda])$. However, $[\Lambda]$ is a non-degenerate antisymmetric bilinear form on $E/\ker \Lambda$ and hence, if $F$ is maximal isotropic in $E$, then $F/\ker \Lambda$ is maximal isotropic in $E/\ker \Lambda$, which is equivalent to $F^{\perp_\Lambda}/\ker \Lambda \overset{(3.4)}{=} (F/\ker \Lambda)^{\perp_{[\Lambda]}} = F/\ker \Lambda$ (see, e.g., [1, Proposition 5.3.3]). Since both $F$ and $F^{\perp_\Lambda}$ contain $\ker \Lambda$, this implies that $F = F^{\perp_\Lambda}$ and hence (3.5) implies (3.3).

Conversely, let $F$ be an isotropic subspace of $E$ such that $\dim F = \dim E - \frac{1}{2} \operatorname{rank} \Lambda$ and $G \supset F$ a maximal isotropic subspace containing $F$. By (3.3), $\dim G = \dim E - \frac{1}{2} \operatorname{rank} \Lambda = \dim F$ which shows that $F = G$, i.e., $F$ itself is maximal isotropic.

If $\ker \Lambda$ were a maximal isotropic subspace, by (3.3), we would have

$$\dim \ker \Lambda = \dim E - \frac{1}{2} \operatorname{rank} \Lambda = \frac{1}{2} \dim E + \frac{1}{2} \dim \ker \Lambda \implies \dim E = \dim \ker \Lambda \implies \Lambda = 0,$$

which is excluded, by hypothesis.

(iv) Let $f_1, \cdots, f_r \in F \setminus \ker\Lambda$ be linearly independent vectors. We now show that the covectors

$$\begin{aligned} &\Lambda(f_1, \cdot), \cdots, \Lambda(f_r, \cdot) \in E^* \text{ are linearly independent} \\ &\qquad \text{if and only if } \operatorname{span}\{f_1, \cdots, f_r\} \cap \ker\Lambda = 0. \end{aligned} \tag{3.6}$$

Indeed, suppose that $\Lambda(f_1, \cdot), \cdots, \Lambda(f_r, \cdot) \in E^*$ are linearly independent and let $g := \alpha_1 f_1 + \cdots + \alpha_r f_r \in \operatorname{span}\{f_1, \cdots, f_r\} \cap \ker\Lambda$, $\alpha_1, \cdots, \alpha_r \in \mathbb{R}$. Then $0 = \Lambda(g, \cdot) = \alpha_1 \Lambda(f_1, \cdot) + \cdots + \alpha_r \Lambda(f_r, \cdot)$ and hence $\alpha_1 = \cdots = \alpha_r = 0$. Thus, $g = 0$, which shows that $\operatorname{span}\{f_1, \cdots, f_r\} \cap \ker\Lambda = 0$. Conversely, if $\operatorname{span}\{f_1, \cdots, f_r\} \cap \ker\Lambda = 0$, then $\alpha_1 \Lambda(f_1, \cdot) + \cdots + \alpha_r \Lambda(f_r, \cdot) = 0$ for $\alpha_1, \cdots, \alpha_r \in \mathbb{R}$, implies $\Lambda(\alpha_1 f_1 + \cdots + \alpha_r f_r, \cdot) = 0$, i.e., $\alpha_1 f_1 + \cdots + \alpha_r f_r \in \ker\Lambda$ and hence $\alpha_1 f_1 + \cdots + \alpha_r f_r = 0$. Since $f_1, \cdots, f_r$ are linearly independent, it follows that $\alpha_1 = \cdots = \alpha_r = 0$ which proves the linear independence of $\Lambda(f_1, \cdot), \cdots, \Lambda(f_r, \cdot)$.

Now, suppose that $F$ is an isotropic subspace of $E$ and that $\dim F = \dim E - r$. By (iii), $F$ is maximal isotropic and $\ker \Lambda \subsetneq F$ (since $\Lambda \neq 0$). Thus

$$\dim F - \dim \ker \Lambda = \dim E - \frac{1}{2} \operatorname{rank} \Lambda - \dim E + \operatorname{rank} \Lambda = \frac{1}{2} \operatorname{rank} \Lambda = r$$

and hence $\dim(F/\ker\Lambda) = r$. Let $\{[f_1], \ldots, [f_r]\}$ be a basis of $F/\ker\Lambda$, so $\{f_1, \ldots, f_r\} \subset F \setminus \ker\Lambda$ are linearly independent. We now show that $\operatorname{span}\{f_1, \cdots, f_r\} \cap \ker\Lambda = 0$. Indeed if $\alpha_1 f_1 + \cdots + \alpha_r f_r \in \ker\Lambda$ for some $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$, it follows that $\alpha_1 [f_1] + \cdots + \alpha_r [f_r] = [0]$, whence $\alpha_1 = \cdots = \alpha_r = 0$. By (3.6), $\Lambda(f_1, \cdot), \cdots, \Lambda(f_r, \cdot) \in E^*$ are also linearly independent. In addition, since $\ker\Lambda \subset F$ and $\operatorname{span}\{f_1, \ldots, f_r\} \cap \ker\Lambda = 0$, it follows that $\operatorname{span}\{f_1, \ldots, f_r\} \oplus \ker\Lambda \subseteq F$. However, $\dim(\operatorname{span}\{f_1, \cdots, f_r\} \oplus \ker\Lambda) = r + \dim\ker\Lambda = \frac{1}{2}\operatorname{rank}\Lambda + \dim E - \operatorname{rank}\Lambda = \dim E - r = \dim F$, by hypothesis. This shows that $\operatorname{span}\{f_1, \ldots, f_r\} \oplus \ker\Lambda = F$.

Conversely, suppose that $F \subset E$ is an isotropic subspace and that there exist $r$ linearly independent vectors $f_1, \ldots, f_r \in F \setminus \ker\Lambda$ such that $\Lambda(f_1, \cdot), \cdots, \Lambda(f_r, \cdot) \in E^*$ are also linearly independent. By (3.6), $\operatorname{span}\{f_1, \ldots, f_r\} \cap \ker\Lambda = 0$. Further, if $\operatorname{span}\{f_1, \cdots, f_r\} \oplus \ker\Lambda = F$, it is trivial to see $\dim F = \dim E - r$. □

**Remark 3.1.** In the statement of Lemma 3.1(iv), condition (2) is essential. Indeed, let $E = \mathbb{R}^{2r+k}$ be equipped with the standard basis $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{2r+k}\}$ and the standard (degenerate) skew-symmetric quadratic form

$$\Lambda = \begin{pmatrix} 0 & \mathsf{E} & 0 \\ -\mathsf{E} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \operatorname{rank}\Lambda = 2r, \quad \dim(\ker\Lambda) = k > 0.$$

For the $r$-dimensional subspace $F = \operatorname{Span}\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_r\}$, condition (1) in Lemma 3.1(iv) holds, but condition (2) is violated because $\dim F = r \neq r + k = \dim E - r$. ♢

**Definition 3.2.** *Let $(M, \mathcal{A})$ be a Poisson manifold. A set $\mathcal{F} \subset \mathcal{C}^\infty(M)$ of commutative functions with respect to the Poisson tensor $\mathcal{A}$ is said to be* complete *if the subspaces $\mathsf{d}\mathcal{F}(x) := \operatorname{span}\{\mathsf{d}f(x) \mid f \in \mathcal{F}\} \subset T_x^*M$ are maximal isotropic (in the sense of Definition 3.1) for $x$ in an open dense subset of $M$.*

**Remark 3.2.** The set $\mathcal{F}$ of functions can be taken in the category of $\mathcal{C}^\infty$, real analytic, polynomial, or complex analytic functions, when $(M, \mathcal{A})$ is $\mathcal{C}^\infty$, real analytic, affine algebraic, or complex analytic manifolds, respectively. ♢

**Proposition 3.2.** *Let $(M, \mathcal{A})$ be a Poisson manifold and $\mathcal{F}$ an involutive set of functions globally defined on $M$ with respect to the Poisson tensor $\mathcal{A}$. Then, $\mathcal{F}$ is complete if and only if $\dim\mathsf{d}\mathcal{F}(x) = \dim M - \frac{1}{2}\operatorname{rank}\mathcal{A}(x)$ for all $x$ in an open dense subset of $M$.*

*Proof.* Before beginning the proof, recall a few elementary facts about Poisson manifolds (see, e.g., [36], [34, Chapter 2], [29, §4.1.11-4.1.34], [23, Chapter 10]). First, $\operatorname{rank}\mathcal{A}(x) = \dim\mathcal{L}$ for any $x \in \mathcal{L}$, where $\mathcal{L}$ is a symplectic leaf of $M$. Second, $M \ni x \mapsto \operatorname{rank}\mathcal{A}(x)$ is lower semicontinuous (the rank cannot decrease in a neighborhood of $x$). The point $x \in M$ is a *regular point* if $x$ is a local maximum of $\operatorname{rank}\mathcal{A}(x)$, which is equivalent to $\operatorname{rank}\mathcal{A}(x') =$ *constant* for all $x'$ in an open neighborhood of $x$. Third, the set of regular points is open and dense, but not connected, in general. Fourth, if $M$ is connected, the set of regular points coincides with the union of all maximal dimensional symplectic leaves of $M$. Fifth, all symplectic leaves are initial submanifolds (see, e.g. [29, §1.1.8-1.1.10]); in particular, the intersection of any open set in $M$ with a symplectic leaf is open in the symplectic leaf, but an arbitrary open set in the leaf is not obtained in this fashion, in general, i.e., the topology of the symplectic leaves is finer than the relative topology induced by the topology of $M$.

Since $\mathcal{F}$ is a commutative set of functions relative to the Poisson bracket, by Definition 3.1, it follows that $\mathsf{d}\mathcal{F}(x)$ is an isotropic subspace of $(T_x^*M, \mathcal{A}(x))$. By Definition 3.2, $\mathcal{F}$ is complete, if and only if the subspaces $\mathsf{d}\mathcal{F}(x) \subset T_x^*M$ are maximal isotropic for $x$ in an open dense subset of $M$, i.e., by Lemma 3.1(iii) for $E = T_x^*M$, $F = \mathsf{d}\mathcal{F}(x)$, and $\Lambda = \mathcal{A}(x)$, if and only if $\dim\mathsf{d}\mathcal{F}(x) = \dim M - \frac{1}{2}\operatorname{rank}\mathcal{A}(x)$ for all $x$ in this open dense subset of $M$. □

Now, we discuss the relation of the completeness of the set $\mathcal{F}$ of functions on the Poisson manifold $(M, \mathcal{A})$ with the complete integrability on each generic symplectic leaves, as well as with the local Casimir functions with respect to the Poisson bracket $\mathcal{A}$. To this end, we introduce the sheaf $\widetilde{\mathcal{F}}$ on $M$ of the germs of convergent power series in the elements of $\mathcal{F}$. The sheaf $\widetilde{\mathcal{F}}$ is given by the presheaf which, to an open subset $V \subset M$, associates the ring of analytic functions $f = f\left(h_1|_V, \ldots, h_n|_V\right)$ in the restriction of finite number of elements $h_1, \ldots, h_n \in \mathcal{F}$ to $V$. The next lemma is a direct consequence of the definitions

**Lemma 3.3.** *If $\mathcal{F}$ is involutive, then, for any open set $V \subset M$, $\widetilde{\mathcal{F}}(V)$ is also involutive.*

Let $\mathsf{d}\widetilde{\mathcal{F}}$ be the sheaf of the germs of differentials $\mathsf{d}f$ of local sections $f$ of $\widetilde{\mathcal{F}}$.

**Lemma 3.4.** *At any point $x \in M$, the stalk $\mathsf{d}\widetilde{\mathcal{F}}_x$ can be identified with $\mathsf{d}\mathcal{F}(x)$. In other words, for any sufficiently small open neighborhood $V \subset M$ of $x$, the set $\left\{\eta(x) \in T_x^*M \,\middle|\, \eta \in \mathsf{d}\widetilde{\mathcal{F}}(V)\right\} = \mathsf{d}\left(\widetilde{\mathcal{F}}(V)\right)(x)$ of values of one-forms at $x$ coincides with $\mathsf{d}\mathcal{F}(x)$.*

*Proof.* By the definition of the sheaf $\widetilde{\mathcal{F}}$, we can take an open neighborhood $V \subset M$ of $x$, such that $\widetilde{\mathcal{F}}(V)$ is the ring of the power series of the form $f = \sum_\alpha c_\alpha \left(h_1 - h_1(x)\right)^{\alpha_1} \cdots \left(h_n - h_n(x)\right)^{\alpha_n}$ which converge uniformly and absolutely on $V$. Here, $h_1, \ldots, h_n \in \mathcal{F}$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is the multi-index, whose components $\alpha_i$ run through all the positive integers and zero for all $i = 1, \ldots, n$. Note that $n$ can be any finite positive integer. Then, the corresponding element $\mathsf{d}f \in \mathsf{d}\widetilde{\mathcal{F}}(V)$ can be written as $\mathsf{d}f = \sum_{i=1}^n \frac{\partial f}{\partial h_i}\mathsf{d}h_i$. This implies that $\mathsf{d}f(x) \in \mathsf{d}\mathcal{F}(x) = \operatorname{span}\{\mathsf{d}h(x) \mid h \in \mathcal{F}\}$. Thus, $\left\{\eta(x) \,\middle|\, \eta \in \mathsf{d}\widetilde{\mathcal{F}}(V)\right\} \subset \mathsf{d}\mathcal{F}(x)$. On the other hand, since $h|_V \in \widetilde{\mathcal{F}}(V)$ for any $h \in \mathcal{F}$, we see that $\mathsf{d}h(x) \in \mathsf{d}\widetilde{\mathcal{F}}(V)$, and hence $\mathsf{d}\mathcal{F}(x) \subset \left\{\eta(x) \,\middle|\, \eta \in \mathsf{d}\widetilde{\mathcal{F}}(V)\right\}$. □

To any two open subsets $W \subset V \subset M$, associate the restriction mappings $r_{W,V} : \widetilde{\mathcal{F}}(V) \ni f \mapsto f|_W \in \widetilde{\mathcal{F}}(W)$ and $r_{W,V} : \mathsf{d}\widetilde{\mathcal{F}}(V) \ni \mathsf{d}f \mapsto (\mathsf{d}f)|_W \in \mathsf{d}\widetilde{\mathcal{F}}(W)$. For $f \in \widetilde{\mathcal{F}}(V)$, $\mathsf{d}f = \sum_{i=1}^n \frac{\partial f}{\partial h_i}\mathsf{d}h_i$, as in the proof of the previous lemma. Since $\mathcal{F}$ consists of the functions globally defined on $M$, we have $\mathsf{d}\left(r_{W,V}(f)\right) = \sum_{i=1}^n r_{W,V}\left(\frac{\partial f}{\partial h_i}\mathsf{d}h_i\right) = r_{W,V}\left(\mathsf{d}f\right)$. Thus, using Definition 3.2, we have the following result.

**Lemma 3.5.** *For two open subsets $W \subset V \subset M$, if $\widetilde{\mathcal{F}}(V)$ is complete with respect to the Poisson tensor $\mathcal{A}|_V$, then so is $\widetilde{\mathcal{F}}(W)$ with respect to $\mathcal{A}|_W$.*

Recall that $\widetilde{\mathcal{F}}(M)$ is the ring generated by the analytic functions in finite elements of $\mathcal{F}$.

**Lemma 3.6.** *$\mathcal{F}$ is complete if and only if $\widetilde{\mathcal{F}}(M)$ is complete.*

*Proof.* Suppose that $\widetilde{\mathcal{F}}(M)$ is complete. For any $x$ in an open dense subset of $M$ where $\mathsf{d}\left(\widetilde{\mathcal{F}}(M)\right)(x) \subset T_x^*M$ is maximally isotropic, let $V$ be an open neighborhood of $x$. By Lemma 3.5, $\widetilde{\mathcal{F}}(V)$ is complete, which means $\mathsf{d}\left(\widetilde{\mathcal{F}}(V)\right)(x) \subset T_x^*M$ is maximal isotropic (see Definition 3.2). However, $\mathsf{d}\mathcal{F}(x) = \mathsf{d}\left(\widetilde{\mathcal{F}}(V)\right)(x)$, by Lemma 3.4, which shows that $\mathsf{d}\mathcal{F}(x)$ is maximal isotropic in $T_x^*M$, thus proving the completeness of $\mathcal{F}$.

Conversely, suppose that $\mathcal{F}$ is complete. So, for any $x$ in an open dense subset of $M$, $\mathsf{d}\mathcal{F}(x) := \text{span}\{\mathsf{d}h(x) \mid h \in \mathcal{F}\}$ is a maximal isotropic subspace of $T_x^*M$. Since $\mathcal{F} \subset \widetilde{\mathcal{F}}(M)$, we have $\mathsf{d}\mathcal{F}(x) \subset \mathsf{d}\left(\widetilde{\mathcal{F}}(M)\right)(x)$. However, $\mathsf{d}\left(\widetilde{\mathcal{F}}(M)\right)(x)$ is isotropic by Lemma 3.3, and hence $\mathsf{d}\mathcal{F}(x) = \mathsf{d}\left(\widetilde{\mathcal{F}}(M)\right)(x)$, which proves the lemma. □

Now, we introduce the two conditions linked to the local existence of Casimir functions and the global existence of the functions which form a completely integrable system on the generic leaf of the Poisson manifold, respectively. We denote the open dense set of all points $x \in M$ where $\mathsf{d}\mathcal{F}(x) \subset T_x^*M$ is maximal isotropic by $U$.

(C1) Around each point $x \in U$, there exist an open neighborhood $V \subset U$ of $x$ and functions $g_1, \ldots, g_k \in \widetilde{\mathcal{F}}(V)$, such that $g_1, \ldots, g_k$ are Casimir functions on $V$ with respect to $\mathcal{A}|_V$, i.e. $\mathcal{A}(\mathsf{d}g_i, \cdot) = 0$ for any $i = 1, \ldots, k$, and $\mathsf{d}g_1(y), \ldots, \mathsf{d}g_k(y) \in T_y^*M$ are linearly independent at any point $y \in V$.

(C2) For any generic symplectic leaf $\mathcal{L} \subset M$, there exist globally defined functions $f_1, \ldots, f_r \in \mathcal{F}$ on $M$ such that $f_1|_{\mathcal{L}}, \ldots, f_r|_{\mathcal{L}}$ are functionally independent on $\mathcal{L}$, i.e. $\mathsf{d}(f_1|_{\mathcal{L}}) \wedge \cdots \wedge \mathsf{d}(f_r|_{\mathcal{L}}) \neq 0$ on an open dense subset of $\mathcal{L}$.

Although the existence of local Casimir functions is guaranteed by Weinstein Splitting Theorem [36], condition (C1) means that these local Casimir functions are included in the ring $\widetilde{\mathcal{F}}(V)$ of analytic functions in the elements of $\mathcal{F}$. The definition of Liouville complete integrability implies the following result.

**Proposition 3.7.** *If condition* (C2) *holds, then the restricted functions $f_1|_{\mathcal{L}}, \ldots, f_r|_{\mathcal{L}}$ form a completely integrable system on $\mathcal{L}$ in the sense of Liouville.*

Next, we show that conditions (C1) and (C2) guarantee local completeness.

**Proposition 3.8.** *If conditions* (C1) *and* (C2) *are satisfied, then, around each point $x \in U$, there exists an open neighborhood $V \subset U$ of $x$ such that $\widetilde{\mathcal{F}}(V)$ is complete with respect to $\mathcal{A}|_V$.*

*Proof.* By condition (C1), around a point $x \in U$, we have an open neighborhood $V \subset U$ of $x$ and Casimir functions $g_1, \ldots, g_k \in \widetilde{\mathcal{F}}(V)$ such that $\mathsf{d}g_1(y), \ldots, \mathsf{d}g_k(y) \in T_y^*M$ are linearly independent at each point $y \in V$. Therefore, $\ker \mathcal{A}(y) = \text{span}\{\mathsf{d}g_1(y), \ldots, \mathsf{d}g_k(y)\}$ at each point $y \in V$. An arbitrary symplectic leaf $\mathcal{L} \cap V$ of $(V, \mathcal{A}|_V)$, where $\mathcal{L}$ is a maximal dimensional symplectic leaf in $(M, \mathcal{A})$, is described as the intersection of the level hypersurfaces of $g_1, \ldots, g_k$ in $V$. By condition (C2), after shrinking $V$ if necessary, there exist $f_1, \ldots, f_r \in \mathcal{F}$ such that $\mathsf{d}(f_1|_{\mathcal{L}})(y), \ldots, \mathsf{d}(f_r|_{\mathcal{L}})(y) \in T_y^*\mathcal{L}$ are linearly independent at each point $y \in V$. Since $T_y^*M = T_y^*\mathcal{L} \oplus \ker \mathcal{A}(y)$ at $y \in V$, the covectors $\mathsf{d}f_1(y), \ldots, \mathsf{d}f_r(y), \mathsf{d}g_1(y), \ldots, \mathsf{d}g_k(y) \in T_y^*M$ are linearly independent. Moreover, the functions $f_1|_V, \ldots, f_r|_V, g_1, \ldots, g_k$ are elements of $\widetilde{\mathcal{F}}(V)$, so that we have $\mathsf{d}\left(\widetilde{\mathcal{F}}(V)\right)(y) = \text{span}\{\mathsf{d}f_1(y), \ldots, \mathsf{d}f_r(y), \mathsf{d}g_1(y), \ldots, \mathsf{d}g_k(y)\}$ at any point $y \in V$. This implies that $\dim\left(\mathsf{d}\left(\widetilde{\mathcal{F}}(V)\right)(y)\right) = r + k$, which is the maximal dimension of isotropic subspaces in $T_y^*M$, at $y \in V$. Therefore, by Lemma 3.1 (iii) and Definition 3.2, $\widetilde{\mathcal{F}}(V)$ is complete with respect to $\mathcal{A}|_V$. □

Conversely, the local completeness implies the condition (C1).

**Proposition 3.9.** *Assume that around each point $x \in U$ there is an open neighborhood $V \subset U$ of $x$ such that $\widetilde{\mathcal{F}}(V)$ is complete with respect to $\mathcal{A}|_V$. Then condition* (C1) *holds.*

*Proof.* By Lemma 3.1 (iii) and Lemma 3.4, we have $\ker \mathcal{A}(y) \subset \mathsf{d}\left(\widetilde{\mathcal{F}}(V)\right)(y) = \mathsf{d}\mathcal{F}(y)$ at any point $y \in V$. Since $\mathsf{d}\left(\widetilde{\mathcal{F}}(V)\right)(y) \subset T_y^*M$ attains the maximal dimension $r+k$, we can, for any sufficiently small open neighborhood $V \subset U$ of $x$, find analytic functions $\xi_1, \ldots, \xi_r$ on $V$ and $h_1, \ldots, h_{r+k} \in \mathcal{F}$ such that $(\xi_1, \ldots, \xi_r, h_1, \ldots, h_{r+k})$ are local coordinates defined on $V$. In this situation, $\mathsf{d}\mathcal{F}(y) = \operatorname{span}\{\mathsf{d}h_1(y), \ldots, \mathsf{d}h_{r+k}(y)\}$ at any point $y \in V$. Shrinking $V$ if necessary, we can further take the analytic one-forms $\omega_1, \ldots, \omega_k \in \Omega^1(V)$ which satisfy $\ker \mathcal{A}(y) = \operatorname{span}\{\omega_1(y), \ldots, \omega_k(y)\}$ at $y \in V$. By the argument of [21, I. 2], the differential system $\omega_1 = 0, \ldots, \omega_k = 0$ is completely integrable in the sense of Frobenius. Thus, shrinking $V$ if necessary, we have analytic functions $g_1, \ldots, g_k$ of the variables $(\xi_1, \ldots, \xi_r, h_1, \ldots, h_{r+k})$ satisfying $\ker \mathcal{A}(y) = \operatorname{span}\{\mathsf{d}g_1(y), \ldots, \mathsf{d}g_k(y)\}$ at any point $y \in V$, where $\mathsf{d}g_1(y), \ldots, \mathsf{d}g_k(y) \in T_y^*M$ are linearly independent. Since $\ker \mathcal{A}(y) \subset \mathsf{d}\mathcal{F}(y)$ for any $y \in V$, we have $\mathsf{d}g_i(y) \in \operatorname{span}\{\mathsf{d}h_1(y), \ldots, \mathsf{d}h_{r+k}(y)\}$ at any point $y \in V$ for $i = 1, \ldots, k$. Indeed, if the function $g_i$ would depend on the variables $(\xi_1, \ldots, \xi_r)$, then we would have $\mathsf{d}g_i(y) \notin \operatorname{span}\{\mathsf{d}h_1(y), \ldots, \mathsf{d}h_{r+k}(y)\} = \mathsf{d}\mathcal{F}(y)$ at some point $y \in V$, which is a contradiction to the fact $\mathsf{d}g_i(y) \in \ker \mathcal{A}(y) \subset \mathsf{d}\mathcal{F}(y)$ at any $y \in V$. Therefore, the functions $g_i$ depend only on $(h_1, \ldots, h_{r+k})$, for $i = 1, \ldots, k$, which means that $g_i \in \widetilde{\mathcal{F}}(V)$. Since $\mathsf{d}g_i(y) \in \ker \mathcal{A}(y)$ at any point $y \in V$, we have $\mathcal{A}(\mathsf{d}g_i, \cdot) = 0$ for $i = 1, \ldots, k$. Hence, $g_i$, $i = 1, \ldots, k$, are Casimir functions with respect to $\mathcal{A}|_V$. □

**Remark 3.3.** In the case of $U(n)$ free rigid body dynamics, since $\mathcal{F}_{\mathsf{J}}$ contains all the Casimir functions $I_0^{(k)}(X) = \dfrac{\left(\sqrt{-1}\right)^k}{k} \operatorname{Tr}\left(X^k\right)$, $\mathcal{F}_{\mathcal{P}} = \mathcal{G}_{\mathsf{J}}$ clearly contains all these globally defined Casimir functions. Note that $\mathcal{G}_{\mathsf{J}} \subset \widetilde{\mathcal{F}_{\mathsf{J}}}(\mathfrak{u}(n))$. ◊

Next, we analyze the relation between the completeness of $\mathcal{F}$ and the complete integrability on generic symplectic leaves of $(M, \mathcal{A})$.

**Proposition 3.10.** *If $\mathcal{F}$ is complete, then condition* (C2) *is satisfied.*

*Proof.* Let $\mathcal{L} \subset M$ be a generic maximal symplectic leaf. Note that $\mathcal{L} \subset U$ and that $\dim \mathcal{L} = 2r$. Take any point $x \in \mathcal{L}$. Since $\mathcal{F}$ is complete, $\mathsf{d}\mathcal{F}(x) \subset T_x^*M$ is maximal isotropic. Then, Lemmas 3.4, 3.5, and Proposition 3.9 imply that there exists an open neighborhood $V \subset U$ of $x$ and local Casimir functions $g_1, \ldots, g_k \in \widetilde{\mathcal{F}}(V)$ with respect to $\mathcal{A}|_V$. In this case, we have $\ker \mathcal{A}(y) = \operatorname{span}\{\mathsf{d}g_1(y), \ldots, \mathsf{d}g_k(y)\}$ at any point $y \in V$. Since $\mathcal{F}$ is complete, Lemma 3.1 (iii) implies that $\ker \mathcal{A}(y) \subset \mathsf{d}\mathcal{F}(y)$. Then, shrinking $V$ if necessary, we can, by means of Lemma 3.1 (iv), take analytic functions $f_1, \ldots, f_r \in \mathcal{F}$ such that $\mathsf{d}\mathcal{F}(y) = \operatorname{span}\{\mathsf{d}f_1(y), \ldots, \mathsf{d}f_r(y), \mathsf{d}g_1(y), \ldots, \mathsf{d}g_k(y)\}$ at any point $y \in V$. Since $T_y^*M = T_y^*\mathcal{L} \oplus \ker \mathcal{A}(y)$ at any point $y \in V \cap \mathcal{L}$, we see that $\mathsf{d}(f_1|_{\mathcal{L}})(y), \ldots, \mathsf{d}(f_r|_{\mathcal{L}})(y) \in T_y^*\mathcal{L}$ are linearly independent at any $y \in V \cap \mathcal{L}$, which is equivalent to $\mathsf{d}(f_1|_{\mathcal{L}})(y) \wedge \ldots \wedge \mathsf{d}(f_r|_{\mathcal{L}})(y) \neq 0$ for any $y \in V \cap \mathcal{L}$. Recall that the functions $f_1, \ldots, f_r$ are analytic and globally defined on $M$. Therefore, $\mathsf{d}(f_1|_{\mathcal{L}}) \wedge \ldots \wedge \mathsf{d}(f_r|_{\mathcal{L}})$ is an analytic $r$-form on $\mathcal{L}$ which cannot vanish on an open set in $\mathcal{L}$. This shows that $\mathsf{d}(f_1|_{\mathcal{L}}), \ldots, \mathsf{d}(f_r|_{\mathcal{L}})$ are linearly independent on an open dense set of $\mathcal{L}$ which proves condition (C2). □

Combining Propositions 3.7 and 3.10, we have the following.

**Proposition 3.11.** *If $\mathcal{F}$ is complete, then, for each maximal dimensional symplectic leaf $\mathcal{L} \subset M$, we can take $r$ functions $f_1, \ldots, f_r \in \mathcal{F}$ such that $f_1|_{\mathcal{L}}, \ldots, f_r|_{\mathcal{L}}$ form a completely integrable system on $\mathcal{L}$ in the sense of Liouville.*

**Remark 3.4.** Real analyticity is also assumed in [8] (pages 435 and 438). On page 441 of [8], the statement preceding Definition 4 is Proposition 3.11. ◊

## 3.2 Complexification

We consider a Poisson manifold $(M, \mathcal{A}_0)$ which admits the bi-Hamiltonian structure induced by the compatible Poisson structure $\mathcal{A}_1$ on $M$ as in Section 2, as well as the associated pencil $\mathcal{P} = \{\lambda_0\mathcal{A}_0 + \lambda_1\mathcal{A}_1 | (\lambda_0 : \lambda_1) \in P_1(\mathbb{R})\}$ of Poisson brackets. We assume that the manifold $M$ and the Poisson structures $\mathcal{A}_0, \mathcal{A}_1$ are considered in the real analytic category, which is obviously true for the $U(n)$ free rigid body dynamics. We use the following codimension two principle to show the completeness of $\mathcal{F}_{\mathcal{P}}$ for the $U(n)$ free rigid body:

**Theorem 3.12** (Bolsinov-Oshemkov [8])**.** *Let $M^{\mathbb{C}}$ be a complexification of the real analytic manifold $M$ and $\mathcal{A}_0^{\mathbb{C}}$, $\mathcal{A}_1^{\mathbb{C}}$ the complexification of $\mathcal{A}_0$, $\mathcal{A}_1$ defined on $M^{\mathbb{C}}$ (see the arguments below). Consider the complex pencil of Poisson tensors $\mathcal{P}^{\mathbb{C}} = \left\{ \lambda_0\mathcal{A}_0^{\mathbb{C}} + \lambda_1\mathcal{A}_1^{\mathbb{C}} \middle| (\lambda_0 : \lambda_1) \in P_1(\mathbb{C}) \right\}$ on $M^{\mathbb{C}}$. Assume that all the complexified tensors $\mathcal{A}_\lambda^{\mathbb{C}} \in \mathcal{P}^{\mathbb{C}}$ on $M^{\mathbb{C}}$, $\lambda \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, have the same rank and that $\mathrm{codim}\mathcal{S}_\lambda^{\mathbb{C}} \geq 2$ for almost all $\lambda \in \widehat{\mathbb{C}}$, where $\mathcal{S}_\lambda^{\mathbb{C}} := \left\{ x \in M^{\mathbb{C}} \middle| \mathrm{rank}\mathcal{A}_\lambda^{\mathbb{C}}(x) < \mathrm{rank}_{\mathbb{C}}\mathcal{P}^{\mathbb{C}} = \mathrm{rank}_{\mathbb{R}}\mathcal{P} \right\}$. Then, $\mathcal{F}_{\mathcal{P}}$ is complete.*

**Complexification of real analytic Poisson manifolds.** Let $M$ be an $n$-dimensional real analytic manifold. An $n$-dimensional complex manifold $X$ is called a *complexification* of $M$ if $M$ is a real analytic submanifold of $X$, regarded as a $2n$-dimensional real analytic manifold, and if, for any point $x \in M$, there exists a neighborhood $\Omega \subset X$ of $x$ and a holomorphic embedding $f : \Omega \to \mathbb{C}^n$ such that $\Omega \cap M = f^{-1}(\mathbb{R}^n)$. This condition is equivalent to say that $M$ is a real analytic submanifold of $X$, regarded as a $2n$-dimensional real analytic manifold, such that $T_xX = T_xM \oplus \sqrt{-1}T_xM$ for all $x \in M$. (See [20, Chapter 1] for a rapid explanation on the complexification of real analytic manifolds.) Of course, this definition of complexification agrees with that for real Lie algebras. In fact, for a real Lie algebra $\mathfrak{g}$, its complexification is given by $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus J\mathfrak{g}$, by means of the complex structure $J$ (multiplication by $\sqrt{-1}$). The Lie bracket $[\cdot, \cdot]^{\mathbb{C}}$ on $\mathfrak{g}^{\mathbb{C}}$ has the expression

$$[X + JY, Z + JW]^{\mathbb{C}} = [X, Z] - [Y, W] + J([X, W] + [Y, Z]),$$

where $X, Y, Z, W \in \mathfrak{g}$ and $[\cdot, \cdot]$ is the Lie bracket of $\mathfrak{g}$.

Assume that we have an $n$-dimensional real analytic Poisson manifold $(M, \mathcal{A})$. Of course, $\mathcal{A}$ is a real analytic section of $TM \wedge TM \to M$. A holomorphic Poisson manifold $(X, \mathcal{B})$, where $\mathcal{B}$ is a holomorphic section of $TX \wedge TX \to X$, is called the *complexification* of $(M, \mathcal{A})$, if $X$ is a complexification of $M$ and if the complexification of $\mathcal{A}(x)$ coincides with $\mathcal{B}(x)$ at any point $x \in M$ in the following sense. The values $\mathcal{A}(x)$ and $\mathcal{B}(p)$ of the Poisson structures $\mathcal{A}$ and $\mathcal{B}$ can be regarded as bilinear forms on the the real cotangent vector space $T_x^*M$ at $x \in M$ and on the holomorphic cotangent vector space $T_p^*X$ at $p \in X$, respectively. At a point $x \in M \subset X$, the holomorphic cotangent vector space to $X$ is decomposed into $T_x^*X = T_x^*M \oplus \sqrt{-1}T_x^*M$. The complexification $\mathcal{A}^{\mathbb{C}}$ on $T_x^*X$ of the real Poisson structure $\mathcal{A}$ is naturally defined at $x \in M$ by

$$\mathcal{A}^{\mathbb{C}}(x)\left(\xi_0 + \sqrt{-1}\xi_1, \eta_0 + \sqrt{-1}\eta_1\right) = \mathcal{A}(x)(\xi_0, \eta_0) - \mathcal{A}(x)(\xi_1, \eta_1) + \sqrt{-1}\left(\mathcal{A}(x)(\xi_0, \eta_1) + \mathcal{A}(x)(\xi_1, \eta_0)\right),$$

where $\xi_0, \xi_1, \eta_0, \eta_1 \in T_x^*M$. We say that the complexification of $\mathcal{A}(x)$ coincides with $\mathcal{B}(x)$ at $x \in M$, if $\mathcal{B}(x) = \mathcal{A}^{\mathbb{C}}(x)$.

On the dual $\mathfrak{g}^*$ of a real Lie algebra $\mathfrak{g}$, we have the Lie-Poisson bracket $\{\cdot, \cdot\}$

$$\{f, g\}(\xi) = \left\langle \xi, \left[\frac{\delta f}{\delta \xi}(\xi), \frac{\delta g}{\delta \xi}(\xi)\right]\right\rangle,$$

where $\xi \in \mathfrak{g}$ and $f, g$ are smooth functions on $\mathfrak{g}^*$; $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. Here, for a smooth function $f$ on $\mathfrak{g}^*$, its functional derivative $\dfrac{\delta f}{\delta \xi} \in \mathfrak{g}$ at $\xi \in \mathfrak{g}^*$ is defined by $\lim_{\epsilon\to 0}\dfrac{f(\xi+\epsilon\delta\xi)-f(\xi)}{\epsilon}=\left\langle \delta\xi, \dfrac{\delta f}{\delta\xi}(\xi)\right\rangle, \delta\xi\in\mathfrak{g}^*$. The duality pairing $\langle\cdot,\cdot\rangle^{\mathbb{C}}$ between the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and its dual $\mathfrak{g}^{\mathbb{C}*}$ is naturally given by

$$\langle \xi+J\eta, X+JY\rangle^{\mathbb{C}}=\langle\xi,X\rangle-\langle\eta,Y\rangle+J\left(\langle\xi,Y\rangle+\langle\eta,X\rangle\right),$$

where $X+JY\in\mathfrak{g}^{\mathbb{C}}, \xi+J\eta\in\mathfrak{g}^{\mathbb{C}*}$. For holomorphic functions $F,G$ on $\mathfrak{g}^{\mathbb{C}*}$, we define

$$\{F,G\}^{\mathbb{C}}(\zeta)=\left\langle \zeta, \left[\frac{\partial F}{\partial\zeta}(\zeta), \frac{\partial G}{\partial\zeta}(\zeta)\right]^{\mathbb{C}}\right\rangle^{\mathbb{C}}, \qquad \zeta\in\mathfrak{g}^{\mathbb{C}*}$$

where $\dfrac{\partial F}{\partial\zeta}(\zeta), \dfrac{\partial G}{\partial\zeta}(\zeta)\in\mathfrak{g}^{\mathbb{C}}$ denotes the holomorphic functional derivative for the holomorphic functions $F,G$, i.e.,

$$\lim_{\epsilon\to 0}\frac{F(\zeta+\epsilon\delta\zeta)-F(\zeta)}{\epsilon}=\left\langle \delta\zeta, \frac{\partial F}{\partial\zeta}(\zeta)\right\rangle^{\mathbb{C}}, \qquad \delta\zeta\in\mathfrak{g}^{\mathbb{C}*}.$$

Here, $\epsilon\in\mathbb{C}$ is a complex number with sufficiently small modulus and the derivation is regarded as complex derivative. Letting $\zeta=\xi+J\eta\in\mathfrak{g}^{\mathbb{C}*}$, such that $\xi,\eta\in\mathfrak{g}^*$, the holomorphic functional derivative has the form $\dfrac{\partial}{\partial\zeta}=\dfrac{1}{2}\left(\dfrac{\delta}{\delta\xi}-J\dfrac{\delta}{\delta\eta}\right)$. Denoting the real and the imaginary parts of the holomorphic function $F$ by $f_0=\mathrm{Re}(F), f_1=\mathrm{Im}(F)$, we have $\dfrac{\delta f_0}{\delta\xi}=\dfrac{\delta f_1}{\delta\eta}, \dfrac{\delta f_0}{\delta\eta}=-\dfrac{\delta f_1}{\delta\xi}$ by means of the Cauchy-Riemann relations. Now, it is easy to check that $\{\cdot,\cdot\}^{\mathbb{C}}$ is exactly the complexification of the Lie-Poisson bracket $\{\cdot,\cdot\}$ on $\mathfrak{g}^*$ in the above sense. In fact, writing the real and the imaginary parts of $G$ by $g_0=\mathrm{Re}(G), g_1=\mathrm{Im}(G)$, we have

$$\begin{aligned}\left[\frac{\partial F}{\partial\zeta}(\zeta), \frac{\partial G}{\partial\zeta}(\zeta)\right]^{\mathbb{C}}&=\frac{1}{4}\left[\frac{\delta f_0}{\delta\xi}(\zeta)+\frac{\delta f_1}{\delta\eta}(\zeta)+J\left(\frac{\delta f_0}{\delta\xi}(\zeta)-\frac{\delta f_1}{\delta\eta}(\zeta)\right),\right.\\&\qquad\left.\frac{\delta g_0}{\delta\xi}(\zeta)+\frac{\delta g_1}{\delta\eta}(\zeta)+J\left(\frac{\delta g_0}{\delta\xi}(\zeta)-\frac{\delta g_1}{\delta\eta}(\zeta)\right)\right]^{\mathbb{C}}\\&=\left[\frac{\delta f_0}{\delta\xi}(\zeta)+J\frac{\delta f_1}{\delta\xi}(\zeta), \frac{\delta g_0}{\delta\xi}(\zeta)+J\frac{\delta g_1}{\delta\xi}(\zeta)\right]^{\mathbb{C}},\end{aligned}$$

by using the Cauchy-Riemann relations. Then, thinking of $\xi\in\mathfrak{g}^*$ as an element of $\mathfrak{g}^{\mathbb{C}*}$, we have

$$\begin{aligned}\{F,G\}^{\mathbb{C}}(\xi)&=\left\langle \xi, \left[\frac{\partial F}{\partial\zeta}(\xi), \frac{\partial G}{\partial\zeta}(\xi)\right]^{\mathbb{C}}\right\rangle^{\mathbb{C}}\\&=\left\langle \xi, \left[\frac{\delta f_0}{\delta\xi}(\xi)+J\frac{\delta f_1}{\delta\xi}(\xi), \frac{\delta g_0}{\delta\xi}(\xi)+J\frac{\delta g_1}{\delta\xi}(\xi)\right]^{\mathbb{C}}\right\rangle^{\mathbb{C}}\\&=\left\langle \xi, \left[\frac{\delta f_0}{\delta\xi}(\xi), \frac{\delta g_0}{\delta\xi}(\xi)\right]-\left[\frac{\delta f_1}{\delta\xi}(\xi), \frac{\delta g_1}{\delta\xi}(\xi)\right]\right\rangle\\&\qquad+J\left(\left\langle \xi, \left[\frac{\delta f_0}{\delta\xi}(\xi), \frac{\delta g_1}{\delta\xi}(\xi)\right]+\left[\frac{\delta f_1}{\delta\xi}(\xi), \frac{\delta g_0}{\delta\xi}(\xi)\right]\right\rangle\right),\end{aligned}$$

which is the complexification of the real analytic Poisson bracket $\{\cdot,\cdot\}$, as defined above.

It is known that complexification of a paracompact real analytic manifold defined as above is unique in the sense that, if $X_1$ and $X_2$ are complexifications of the paracompact real analytic manifold $M$, then there is another complexification $X_0$ of $M$ such that $X_0$ is open (or can holomorphically be embedded as open subsets) in both $X_1$ and $X_2$ (cf. [9]). It is further proved by Grauert in [15] that every real analytic manifold has a complexification which is a Stein manifold. As for Poisson manifolds, we will show below the existence and uniqueness of the complexification of any paracompact real analytic Poisson manifold. Note that Theorem 3.12 by Bolsinov and Oshemkov [8] is based on the assumption that there exists a complexification of the concerned real Poisson manifold.

We now prove the existence and uniqueness of the complexification of a given paracompact real analytic Poisson manifold $(M, \mathcal{A})$. By the above mentioned facts, there is a complexification $X$ of the real analytic manifold $M$. Then, each point $p \in M$, has a neighborhood $\Omega_p \subset X$ with holomorphic coordinates $(z^1 = x^1 + \sqrt{-1}y^1, \dots, z^n = x^n + \sqrt{-1}y^n)$ on $\Omega_p$ such that $M \cap \Omega_p = \{y^1 = \cdots = y^n = 0\}$ and that $p$ is the origin $(z^1, \dots, z^n) = (0, \dots, 0)$ at $p$. Note that $(x^1, \dots, x^n)$ are real analytic coordinates of $M$ on $M \cap \Omega_p$ with the origin $(x^1, \dots, x^n) = (0, \dots, 0)$ at $p$. Now, the Poisson structure $\mathcal{A}$ on $M$ can be written as

$$\mathcal{A}(q) = \sum_{i,j=1}^{n} A^{ij}(x^1, \cdots, x^n) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$$

at $q = (x^1, \dots, x^n)$ in a sufficiently small neighborhood $|x^i| < \epsilon$, $i = 1, \dots, n$. In this neighborhood, we can assume that the analytic functions $A^{ij}(x^1, \cdots, x^n)$ can be given as a convergent power series

$$A^{ij}(x^1, \cdots, x^n) = \sum_{i_1, \cdots, i_n = 0}^{n} a^{ij}_{i_1, \cdots, i_n} (x^1)^{i_1} \cdots (x^n)^{i_n}.$$

Since $\mathcal{A}$ is a Poisson structure, the coefficients are skew-symmetric, i.e.,

$$a^{ij}_{i_1, \cdots, i_n} = -a^{ji}_{i_1, \cdots, i_n},$$

for any $i_1, \dots, i_n = 0, 1, 2, \dots$ and $i, j = 1, \dots, n$. The vanishing of the Schouten-Nijenhuis tensor:

$$\sum_{i=1}^{n} \left( A^{ij} \frac{\partial A^{kl}}{\partial x^i} + A^{ik} \frac{\partial A^{lj}}{\partial x^i} + A^{il} \frac{\partial A^{jk}}{\partial x^i} \right) = 0, \qquad j, k, l = 1, \cdots, n,$$

implies a bunch of quadratic relations among the coefficients $a^{ij}_{i_1, \cdots, i_n}$. We introduce the bi-vector

$$\mathcal{B}(q) = \sum_{i,j=1}^{n} B^{ij}(z^1, \cdots, z^n) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j},$$

$$B^{ij}(z^1, \cdots, z^n) := \sum_{i_1, \cdots, i_n = 0}^{n} a^{ij}_{i_1, \cdots, i_n} (z^1)^{i_1} \cdots (z^n)^{i_n}, \tag{3.7}$$

in $T_qX \wedge T_qX$ where $q = (z^1, \dots, z^n)$ is in the polydisc $D_p := \{(z^1, \dots, z^n) \in \mathbb{C}^n \mid |z^i| < \epsilon$, $i = 1, \dots, n\}$. Then, this power series converges absolutely on this polydisc and its skew-symmetry is obvious. The Schouten-Nijenhuis tensor for $\mathcal{B}$ vanishes, since the conditions posed on the coefficients $a^{ij}_{i_1, \cdots, i_n}$ are exactly the same as those for the vanishing of the Schouten-Nijenhuis tensor of the real analytic Poisson structure $\mathcal{A}$. Now, setting $D := \bigcup_{p \in M} D_p \subset X$, we easily see that the $(2, 0)$ tensor field $\mathcal{B}$ is well-defined on $D$, which is clearly a holomorphic Poisson structure. (When we patch the two local expressions of the holomorphic Poisson bi-vectors as in (3.7), the transition functions can be obtained from the transition functions for the real analytic Poisson structure $\mathcal{A}$

on $M$, just by replacing formally the analytic coordinates $(x^1,\dots,x^n)$ by $(z^1,\dots,z^n)$.) It is clear that $(D,\mathcal{B})$ is the complexification of $(M,\mathcal{A})$. Therefore, the existence of the complexification of a real analytic Poisson manifold is proved.

The uniqueness follows immediately. In fact, if there is a complexification $(X,\mathcal{B})$ of the real Poisson manifold $(M,\mathcal{A})$, the coefficients of $\mathcal{B}$ must have the power series expansion at $q$ in a small polydisc neighborhood of a point $p\in M$, which is exactly the same as (3.7). (Otherwise, it does not coincide with $\mathcal{A}$ on $M$.) Since holomorphic functions which are equal in an open set coincide everywhere in the domain where they are defined, the uniqueness follows.

## 3.3 Integrability criterion applied to the $\mathfrak{u}(n)$-Euler equation

We examine the two conditions of Theorem 3.12 for the $U(n)$ free rigid body. Although the statement of the theorem is about all the complex parameters $\lambda\in\widehat{\mathbb{C}}$, we check here only the conditions for real $\lambda$'s, since the singular loci $\lambda$ of the pencil $\mathcal{P}$ of the Poisson structures are all real and since the arguments for complex $\lambda$'s are essentially the same as for the real $\lambda$'s. So, let

$$\mathcal{S}_\lambda := \left\{ X\in\mathfrak{u}(n)\,|\,\mathrm{rank}\left(\{\cdot,\cdot\}_{\mathsf{E}+\lambda\mathsf{J}^2}\right)(X) < \mathrm{rank}\mathcal{P}\right\},$$

where $\mathrm{rank}\mathcal{P} = \max_{\lambda\in\mathbb{R}}\left(\mathrm{rank}\left(\{\cdot,\cdot\}_{\mathsf{E}+\lambda\mathsf{J}^2}\right)\right), \mathrm{rank}\left(\{\cdot,\cdot\}_{\mathsf{A}}\right) = \max_{X\in\mathfrak{u}(n)}\mathrm{rank}\left(\{\cdot,\cdot\}_{\mathsf{A}}\right)(X)$, as in Subsection 2.2. Denote by $S$ the linear space underlying the Lie algebra $\mathfrak{u}(n)$ (just forget the Lie algebra structure).

**(i)** We first show that all the brackets $\{\cdot,\cdot\}_{\mathsf{E}+\lambda\mathsf{J}^2}$, $\lambda\in\mathbb{R}\cup\{\infty\}$, have rank $n^2-n$. If $\mathsf{E}+\lambda\mathsf{J}^2$ is non-degenerate, the proof is rather easy. Set $\mathsf{A}=\mathsf{E}+\lambda\mathsf{J}^2$ and consider the linear isomorphism $\Psi_{\mathsf{A}}: S\ni X\mapsto\sqrt{\mathsf{A}}X\sqrt{\mathsf{A}}\in\mathfrak{u}(p,q)$, where $(p,q)$ is the signature of $\mathsf{A}=\mathsf{E}+\lambda\mathsf{J}^2$. Since $\Psi_{\mathsf{A}}$ is a Lie algebra isomorphism between $\mathfrak{g}_{\mathsf{A}}$ and $\mathfrak{u}(p,q)$, by Proposition 2.5, and since $\mathrm{rank}\left(\mathfrak{u}(p,q)\right)=n$, we conclude that $\{\cdot,\cdot\}_{\mathsf{A}}$ has rank $n^2-n$. Note that the rank of the Lie-Poisson bracket $\{\cdot,\cdot\}_{\mathsf{A}}$ is equal to $\dim\mathfrak{g}_{\mathsf{A}}-\mathrm{rank}\mathfrak{g}_{\mathsf{A}}$ , which is $n^2-n$ for $\mathfrak{g}_{\mathsf{A}}\cong\mathfrak{u}(p,q)$, $p+q=n$.

When $\mathsf{A}=\mathsf{E}+\lambda\mathsf{J}^2$ is degenerate, we can assume that $\mathsf{A}=\mathrm{diag}\left(a_1,\dots,a_{n-1},0\right)$, $a_1,\dots,a_{p'}<0$, $a_{p'+1},\dots,a_{p'+q'}>0$, $p'+q'=n-1$, and consider the linear mapping

$$\Psi'_{\mathsf{A}}: S\ni X\mapsto\mathrm{diag}\left(\sqrt{a_1},\dots,\sqrt{a_{n-1}},1\right)X\,\mathrm{diag}\left(\sqrt{a_1},\dots,\sqrt{a_{n-1}},1\right)\in\mathfrak{u}\left(p',q'\right)_{\mathsf{E}'},$$

where $\mathfrak{u}\left(p',q'\right)_{\mathsf{E}'} = \{X\in\mathbb{C}^{n\times n}\,|\,\mathsf{E}_{p',q'+1}X^*+X\mathsf{E}_{p',q'+1}=0\}$, is regarded as a Lie algebra with respect to the degenerate bracket $[\cdot,\cdot]_{\mathsf{E}'}$, $\mathsf{E}'=\mathrm{diag}(\underbrace{1,\dots,1}_{n-1},0)$.

**Lemma 3.13.** *The linear mapping $\Psi'_{\mathsf{A}}$ is a Lie algebra isomorphism between $\mathfrak{g}_{\mathsf{A}}$ and $\mathfrak{u}\left(p',q'\right)_{\mathsf{E}'}$.*

*Proof.* It suffices to verify $[\Psi'_{\mathsf{A}}(X),\Psi'_{\mathsf{A}}(Y)]_{\mathsf{E}'}=\Psi'_{\mathsf{A}}\left([X,Y]_{\mathsf{A}}\right)$, $X,Y\in S$. Let

$$X=\begin{bmatrix}X_{11} & X_{12}\\ X_{21} & X_{22}\end{bmatrix},\qquad Y=\begin{bmatrix}Y_{11} & Y_{12}\\ Y_{21} & Y_{22}\end{bmatrix}$$

be skew-Hermitian matrices, where $X_{11},Y_{11}\in\mathbb{C}^{n\times n}$, $X_{12},Y_{12}\in\mathbb{C}^{n\times 1}$, $X_{21},Y_{21}\in\mathbb{C}^{1\times n}$, and $X_{22},Y_{22}\in\mathbb{C}$. Then

$$\Psi'_{\mathsf{A}}(X)=\begin{bmatrix}\sqrt{\mathsf{A}'}X_{11}\sqrt{\mathsf{A}'} & \sqrt{\mathsf{A}'}X_{12}\\ X_{21}\sqrt{\mathsf{A}'} & X_{22}\end{bmatrix},\quad\text{where}\quad \mathsf{A}'=\mathrm{diag}\left(a_1,\dots,a_{n-1}\right).$$

If

$$Z=\begin{bmatrix}Z_{11} & Z_{12}\\ Z_{21} & Z_{22}\end{bmatrix},\ W=\begin{bmatrix}W_{11} & W_{12}\\ W_{21} & W_{22}\end{bmatrix}\quad\Longrightarrow\quad [Z,W]_{\mathsf{E}'}=\begin{bmatrix}Z_{11}W_{11}-W_{11}Z_{11} & Z_{11}W_{12}-W_{11}Z_{12}\\ Z_{21}W_{11}-W_{21}Z_{11} & Z_{21}W_{12}-W_{21}Z_{12}\end{bmatrix}$$

and hence

$$[\Psi'_{\mathsf{A}}(X), \Psi'_{\mathsf{A}}(Y)]_{\mathsf{E}'} = \begin{bmatrix} \sqrt{\mathsf{A}'}\,(X_{11}\mathsf{A}'Y_{11} - Y_{11}\mathsf{A}'X_{11})\,\sqrt{\mathsf{A}'} & \sqrt{\mathsf{A}'}\,(X_{11}\mathsf{A}'Y_{12} - Y_{11}\mathsf{A}'X_{12}) \\ (X_{21}\mathsf{A}'Y_{11} - Y_{21}\mathsf{A}'X_{11})\,\sqrt{\mathsf{A}'} & X_{21}\mathsf{A}'Y_{12} - Y_{12}\mathsf{A}'X_{12} \end{bmatrix}.$$

On the other hand, we have

$$[X, Y]_{\mathsf{A}} = \begin{bmatrix} X_{11}\mathsf{A}'Y_{11} - Y_{11}\mathsf{A}'X_{11} & X_{11}\mathsf{A}'Y_{12} - Y_{11}\mathsf{A}'X_{12} \\ X_{21}\mathsf{A}'Y_{11} - Y_{21}\mathsf{A}'X_{11} & X_{21}\mathsf{A}'Y_{12} - Y_{21}\mathsf{A}'X_{12} \end{bmatrix},$$

which shows that $[\Psi'_{\mathsf{A}}(X), \Psi'_{\mathsf{A}}(Y)] = \Psi'_{\mathsf{A}}([X, Y])$. □

We now show that the rank of $\mathfrak{u}\,(p', q')_{\mathsf{E}'}$ is $n$. Let $X \in \mathfrak{u}\,(p', q')_{\mathsf{E}'}$ be a generic fixed matrix. By Duflo-Vergne theorem (cf. [23, p.325, §9.3.10] or [11]), the Cartan subalgebra $\mathfrak{h}_X$ containing $X$ is $\mathfrak{h}_X = \{Y \in \mathfrak{u}\,(p', q')_{\mathsf{E}'} \,|\, [X, Y]_{\mathsf{E}'} = 0\}$. It suffices to show that $\dim\mathfrak{h}_X = n$ for generic $X \in \mathfrak{u}\,(p', q')_{\mathsf{E}'}$. As before, set $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$, $Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$, where $X_{11}, Y_{11} \in \mathbb{C}^{(n-1)\times(n-1)}$, $X_{12}, Y_{12} \in \mathbb{C}^{(n-1)\times 1}$, $X_{21}, Y_{21} \in \mathbb{C}^{1\times(n-1)}$, $X_{22}, Y_{22} \in \mathbb{C}$. We consider those $Y$ such that

$$[X, Y]_{\mathsf{E}'} = \begin{bmatrix} X_{11}Y_{11} - Y_{11}X_{11} & X_{11}Y_{12} - Y_{11}X_{12} \\ X_{21}Y_{11} - Y_{21}X_{11} & X_{21}Y_{12} - Y_{21}X_{12} \end{bmatrix} = 0.$$

Note that $\mathsf{E}_{p',q'}X_{11}^* + X_{11}\underline{\mathsf{E}_{p',q'}} = 0$, $X_{12} + \mathsf{E}_{p',q'}X_{21}^* = 0$, $X_{22} + \overline{X_{22}} = 0$, and $\mathsf{E}_{p',q'}Y_{11}^* + Y_{11}\mathsf{E}_{p',q'} = 0$, $Y_{12} + \mathsf{E}_{p',q'}Y_{21}^* = 0$, $Y_{22} + \overline{Y_{22}} = 0$. The matrix $Y$ has to satisfy the following equations:

$$X_{11}Y_{11} - Y_{11}X_{11} = 0, \qquad X_{11}Y_{12} - Y_{11}X_{12} = 0, \qquad X_{21}Y_{12} - Y_{21}X_{12} = 0. \tag{3.8}$$

The second equation is equivalent to $X_{21}Y_{11} - Y_{21}X_{11} = 0$. From the first equation of (3.8), we see that $Y_{11}$ is an arbitrary element of the Cartan subalgebra containing $X_{11}$ in $\mathfrak{u}\,(p', q')$. The dimension of this Cartan subalgebra is $n - 1$. Further, we can assume that $X_{11}$ is invertible, since this holds generically. Then, the second equation of (3.8) implies $Y_{12} = X_{11}^{-1}Y_{11}X_{12}$, which determines $Y_{12}$ as a function of $Y_{11}$. Moreover, the third equation of (3.8) can be deduced from the first two. Note that there is no condition imposed on $Y_{22}$. Thus, the solution space of the equation $[X, Y]_{\mathsf{E}'} = 0$ has dimension $(n - 1) + 1 = n$. This shows that $\dim\mathfrak{h}_X = n$.

**(ii)** Next, we show that $\mathrm{codim}\mathcal{S}_\lambda \geq 2$ for almost all $\lambda \in \mathbb{R} \cup \{\infty\}$. We consider the set of singular elements in $\mathfrak{u}(p, q)$. Any element $X \in \mathfrak{u}(p, q)$ is conjugate to a diagonal matrix $\mathsf{D}$ by the action of $U(p, q)$: $X = g^{-1}\mathsf{D}g$, $g \in U(p, q)$. This is clear, since the diagonal matrices form the standard Cartan subalgebra in $\mathfrak{u}(p, q)$ (see, e.g., [16, §16]) and since any element in $\mathfrak{u}(p, q)$ is conjugate to one of the elements in the standard Cartan subalgebra. Thus, the orbits in $\mathfrak{u}(p, q)$ can be given as

$$\mathcal{O}_{\sqrt{-1}\mathrm{diag}(x_1,\ldots,x_n)} := \left\{ \mathrm{Ad}_g \left( \sqrt{-1}\mathrm{diag}\,(x_1, \ldots, x_n) \right) \middle|\, g \in U(p, q) \right\}.$$

Let $S_1$ be the disjoint union of all orbits $\mathcal{O}_{\sqrt{-1}\mathrm{diag}(x_1,\ldots,x_n)}$, where the components of $(x_1, \ldots, x_n) \in \mathbb{R}^n$ are all distinct. Let $S_j$ be the disjoint union of all orbits $\mathcal{O}_{\sqrt{-1}\mathrm{diag}(x_1,\ldots,x_n)}$, where precisely $j$ of $x_i$'s are equal and there are no additional equalities among the components. Then, $\mathfrak{u}(p, q) = \bigsqcup_{j=1,\ldots,n} S_j$. Further, the dimension of $\mathcal{O}_{\sqrt{-1}\mathrm{diag}(x_1,\ldots,x_n)}$, where exactly $j$ of $x_i$'s are equal and there are no additional equalities, is $n^2 - n - (j^2 - j)$, so that $\dim S_j = n^2 - j^2 + 1$, since the number of parameters for such orbits is $n - j + 1$. The dimension of the orbit can be calculated by recalling the standard diffeomorphism $\mathcal{O}_{\sqrt{-1}\mathrm{diag}(x_1,\ldots,x_n)} \cong U(p, q)/U(r, s) \times \underbrace{U(1) \times \cdots \times U(1)}_{n-j}$, where $r+s=j$, and from $\dim U(r,s)=j^2$. The set $\mathcal{S}_\lambda$ of singular elements of $\mathsf{A}=\mathsf{E}+\lambda\mathsf{J}^2$ is $\bigsqcup_{j=2,\ldots,n} S_j$, which has hence codimension $n^2-(n^2-2^2+1)=3$. This ends the proof of the codimension condition.

Thus, Theorem 3.12 and Proposition 3.11 yield the following result.

**Theorem 3.14.** *The set $\mathcal{F}_{\mathcal{P}}$, or equivalently $\mathcal{G}_{\mathsf{J}}$, of first integrals for the $U(n)$ free rigid body dynamics is complete. The Hamiltonian system induced on each generic orbit, consisting of invertible skew-Hermitian matrices with distinct eigenvalues, is completely integrable.*

**Remark 3.5.** This theorem also follows from a more general result of Brailov, described in [14]. We introduce the necessary terminology. Given a Lie algebra $\mathfrak{g}$, the *ring of invariants* of $\mathfrak{g}$ is defined to be the ring $\mathcal{I}(\mathfrak{g}^*)$ of Casimir functions for the Lie-Poisson bracket on $\mathfrak{g}^*$. If $\rho:\mathfrak{k}\to\mathrm{End}(V)$ is a representation of a Lie algebra $\mathfrak{k}$ on a vector space $V$, the *dual* or *contragredient representation* of $\mathfrak{k}$ on $V^*$ is defined by $X\cdot x:=-\rho(X)^*x$, for any $X\in\mathfrak{k}$ and $x\in V^*$. The *symmetry* or *isotropy subalgebra* of an element $x\in V^*$ is the Lie subalgebra $\mathfrak{k}^x:=\{X\in\mathfrak{k}\mid\rho(X)^*x=0\}\subset\mathfrak{k}$. An element $y\in V^*$ is said to be *in general position* if $\rho^*(\mathfrak{k})\,y:=\{-\rho(Y)^*y\mid Y\in\mathfrak{k}\}$ attains maximal dimension, i.e., $\dim\rho^*(\mathfrak{k})\,y=\max_{x\in V^*}\dim\rho^*(\mathfrak{k})\,x$, or, equivalently, if $\dim\mathfrak{k}^y=\min_{x\in V^*}\dim\mathfrak{k}^x$. If the Lie algebra representation is induced by a representation of an underlying Lie group $K$, then $y\in V^*$ is in general position if and only if the dimension of the $K$-orbit through $y$ is maximal dimensional among all orbits of the contragredient $K$-representation on $V^*$. In particular, if $V=\mathfrak{k}$ and $\rho$ is the adjoint representation, the elements in general position in $\mathfrak{k}^*$ are those whose coadjoint orbits are the generic symplectic leaves of the Lie-Poisson structure on $\mathfrak{k}^*$. The codimension of these maximal dimensional coadjoint orbits in $\mathfrak{k}^*$ is called the *index* of the Lie algebra $\mathfrak{k}$ and is denoted by $\mathrm{ind}\,\mathfrak{k}$.

With these preparatory remarks, Theorem 20.4, p.221, of [14] can be formulated in the following manner.

**Theorem 3.15** (Brailov)**.** *Let $\mathfrak{k}$ be a Lie algebra and $\rho:\mathfrak{k}\to\mathrm{End}(V)$ a representation of $\mathfrak{k}$ on a vector space $V$. Let $\mathfrak{g}:=\mathfrak{k}\ltimes V$ be the semidirect product of $\mathfrak{k}$ with $V$. Assume the following two conditions:*

1. *The number of independent polynomial invariants of $\mathfrak{g}$ (i.e., the polynomial Casimir functions on the dual space $\mathfrak{g}^*$ with respect to the Lie-Poisson bracket) is equal to the index of the Lie algebra $\mathfrak{g}$.*

2. *For an arbitrary element $y\in V^*$ in general position, the number of independent invariant polynomials in $\mathcal{I}(\mathfrak{k}^{y*})$ is equal to the index $\mathrm{ind}(\mathfrak{k}^y)$ of $\mathfrak{k}^y$. Further, for an arbitrary element $a'\in\mathfrak{k}^{y*}$ in general position, the ring*

$$\mathcal{I}_{a'}(\mathfrak{k}^{y*}):=\{f_\lambda\,|\,f_\lambda(x)=f(x+\lambda a'),x\in\mathfrak{k}^{y*},\lambda\in\mathbb{R},f\in\mathcal{I}(\mathfrak{k}^{y*})\}$$

*of functions is involutive and complete relative to the Lie-Poisson structure of $\mathfrak{k}^{y*}$.*

*Then, for an arbitrary element $a\in\mathfrak{g}^*$ in general position, the set $\mathcal{I}_a(\mathfrak{g}^*)\cup V$ of functions on $\mathfrak{g}^*$ is involutive and complete with respect to the Lie-Poisson bracket on $\mathfrak{g}^*$, where $\mathcal{I}_a(\mathfrak{g}^*):=\{f_\lambda\mid f_\lambda(x):=f(x+\lambda a),\,x\in\mathfrak{g}^*,\,\lambda\in\mathbb{R},\,f\in\mathcal{I}(\mathfrak{g}^*)\}$ and $V=V^{**}$ is regarded as the set of all linear functionals on $V^*$, extended to $\mathfrak{g}^*$, by means of the inclusion $V\ni v\mapsto(0,v)\in\mathfrak{g}$.*

For $\mathfrak{g}=\mathfrak{u}(n)$, we take $\mathfrak{k}=\mathfrak{su}(n)$, $V=\sqrt{-1}\mathbb{R}$, and $\rho$ the trivial representation. The two conditions in Brailov's Theorem hold. Indeed, since $\mathrm{Tr}\left(X^k\right)$, $k=1,\ldots,n$, are independent invariant polynomials and since $\mathrm{rank}\,(\mathfrak{u}(n))=n$, the first condition is satisfied. Since $\rho$ is trivial, it follows that $\mathfrak{k}^y=\mathfrak{k}$, for all $y\in V^*$. Since $\mathfrak{k}^*\cong\mathfrak{k}=\mathfrak{su}(n)$, $\mathfrak{k}^*$ has $n-1$ invariant polynomials, the number of which equals the rank of $\mathfrak{su}(n)$. Since $\mathfrak{su}(n)$ is a simple Lie algebra, by a result of Mishchenko and Fomenko [26], for any element $a' \in \mathfrak{h}^* = \mathfrak{su}(n)^*$ in general position, the ring of functions $\mathcal{I}_{a'}\left(\mathfrak{su}(n)^*\right)$ has all its elements Poisson commuting (relative to the Lie-Poisson bracket) and is complete. Brailov's theorem implies hence that for any generic element $a \in \mathfrak{u}(n)^*$, $\mathcal{I}_a\left(\mathfrak{u}(n)^*\right) \cup V = \mathcal{I}_a\left(\mathfrak{su}(n)^*\right) \cup V = \mathcal{I}_a\left(\mathfrak{u}(n)^*\right)$ is involutive and complete. $\Diamond$

# 4 Equilibria of the $\mathfrak{u}(n)$ free rigid body

In this section, we consider the equilibria of the $U(n)$ free rigid body and their non-degeneracy. We begin with some general considerations.

**Definition 4.1** (Bolsinov-Oshemkov [8]). *Let $(M, \mathcal{A})$ be a Poisson manifold and $\mathcal{F} \subset \mathcal{C}^\infty(M)$ a set of commutative functions with respect to the Poisson tensor $\mathcal{A}$ which is complete (see Definition 3.2). The* common equilibrium points *of $\mathcal{F}$ is the set $\{m \in M \mid \Xi_f(m) = 0, \forall f \in \mathcal{F}\}$.*

Let $f \in \mathcal{F}$. Note that any isolated equilibrium of the Hamiltonian vector field $\Xi_f$ is a common equilibrium of $\mathcal{F}$. Indeed, if $m \in M$ is an isolated equilibrium of $\Xi_f$ and if there would exist some $g \in \mathcal{F}$ such that $\Xi_g(m) \neq 0$, then $\left\{ \phi^t_{\Xi_g}(m) \,\middle|\, |t| < \epsilon \right\} \ni m$ would be a continuous curve consisting of distinct points for sufficiently small $\epsilon > 0$ where $\phi^t_{\Xi_g}$ is the flow of $\Xi_g$. Since $[\Xi_f, \Xi_g] = -\Xi_{\{f,g\}} = 0$, it follows that $\Xi_f\left(\phi^t_{\Xi_g}(m)\right) = T_m\phi^t_{\Xi_g}\left(\Xi_f(m)\right) = 0$, which would mean that $\left\{ \phi^t_{\Xi_g}(m) \,\middle|\, |t| < \epsilon \right\}$ is a curve consisting of equilibria of the Hamiltonian vector field $\Xi_f$. This is a contradiction since $m$ is an isolated equilibrium of $\Xi_f$.

**Theorem 4.1** (Bolsinov-Oshemkov [8]). *Let $\mathcal{P} := \{\mathcal{A}_\lambda = \mathcal{A}_0 + \lambda\mathcal{A}_1 \mid \lambda \in \mathbb{R}\}$ be a pencil of Poisson brackets given by two compatible Poisson tensors $\mathcal{A}_0$ and $\mathcal{A}_1$ on the manifold $M$. Let $\mathcal{F}_\mathcal{P}$ be the commutative ring generated by the Casimir functions for generic Poisson tensors $\mathcal{A}_\lambda \in \mathcal{P}$, i.e. for those Poisson tensors $\mathcal{A}_\lambda \in \mathcal{P}$ which satisfy* $\operatorname{rank}\mathcal{A}_\lambda = \operatorname{rank}\mathcal{P}$. *A point $m \in M$ is a common equilibrium point of $\mathcal{F}_\mathcal{P}$ if and only if* $\ker \mathcal{A}_\lambda(m) = \ker \mathcal{A}_\mu(m)$ *for all $\lambda, \mu \in \mathbb{R}$.*

We apply this criterion to the $U(n)$ free rigid body. From now on, in the rest of the paper, we *assume that the Hermitian matrix* $\mathsf{J}$ in the definition of the inertia tensor for the $U(n)$ free rigid body *is diagonal with distinct entries.* Let $\mathcal{O} := \left\{ g\left(\sqrt{-1}\operatorname{diag}\left(x_1, \ldots, x_n\right)\right) g^* \,\middle|\, g \in U(n) \right\} \subset \mathfrak{u}(n)$ be a generic orbit, i.e., all $x_i$ are distinct and $x_i \neq 0$, $i = 1, \ldots, n$.

**Theorem 4.2.** *On a generic orbit $\mathcal{O} \subset \mathfrak{u}(n)$, the set of common equilibrium points of $\mathcal{F}_\mathcal{P}$ is $\mathfrak{h}_0 \cap \mathcal{O}$.*

*Proof.* Let $\mathfrak{h}_0$ denote the Cartan subalgebra consisting of diagonal matrices in $\mathfrak{u}(n)$. It suffices to show that the kernels of the Poisson brackets $\{\cdot,\cdot\}(X)$ and $\{\cdot,\cdot\}_{\mathsf{J}^2}(X)$ coincide if and only if $X \in \mathfrak{h}_0$ for a generic $X \in \mathfrak{u}(n)$. Here, $X$ is called *generic* if the eigenvalues of $X$ are distinct and different from zero. The kernels of $\{\cdot,\cdot\}(X)$ and $\{\cdot,\cdot\}_{\mathsf{J}^2}(X)$ are given by the Cartan subalgebras $\mathfrak{h}_X = \{Y \in \mathfrak{u}(n) \mid [X, Y] = 0\}$ and $\{Y \in \mathfrak{u}(n) \mid [X, Y]_{\mathsf{J}^2} = 0\}$, respectively. Recall that $\dim \mathfrak{h}_X = n$ and that $[X, Y]_{\mathsf{J}^2} := X\mathsf{J}^2Y - Y\mathsf{J}^2X$ (see §2.3). Since $\mathsf{J}^2$ is diagonal, it is obvious that these Cartan subalgebras coincide if $X$ is diagonal. Thus, the points in $\mathfrak{h}_0$ are common equilibria.

So, all that remains to be shown is that if a generic $X \in \mathfrak{u}(n)$ is a common equilibrium point of $\mathcal{F}_\mathcal{P}$, then it is diagonal. Let $X$ be such an element. Then, Theorem 4.1 implies $[X, Y] = 0$, $[X, Y]_{\mathsf{J}^2} = 0$, i.e., $XY = YX$, $X\mathsf{J}^2Y = Y\mathsf{J}^2X$, for all $Y \in \mathfrak{h}_X$. Since $X$ is invertible, we have $X^{-1}Y = YX^{-1}$, $X^{-1}Y\mathsf{J}^2 = \mathsf{J}^2YX^{-1}$, so that $X^{-1}Y\mathsf{J}^2 = \mathsf{J}^2X^{-1}Y$, $YX^{-1}\mathsf{J}^2 = \mathsf{J}^2YX^{-1}$. Because $\mathsf{J}^2$ is diagonal with distinct entries, this implies that $\mathsf{D} := X^{-1}Y = YX^{-1}$ is a diagonal matrix. Since $X^* = -X$ and $Y^* = -Y$, we have $\mathsf{D}^* = (X^{-1}Y)^* = Y^*(X^{-1})^* = YX^{-1} = \mathsf{D}$, i.e. $\mathsf{D}$ is a real diagonal matrix. Thus, $\mathfrak{h}_X$ is a subspace of the vector space $\mathcal{D}_X := \{Y = \mathsf{D}X = X\mathsf{D} \mid \mathsf{D} \text{ is real diagonal}\}$. However, $\dim \mathcal{D}_X = n = \dim \mathfrak{h}_X$ implies $\mathfrak{h}_X = \mathcal{D}_X$.

In particular, if $\mathsf{D}$ is a diagonal matrix with distinct entries, the condition $\mathsf{D}X = X\mathsf{D}$ implies that $X$ is diagonal, i.e., $X \in \mathfrak{h}_0$. Thus, the generic common equilibrium points of $\mathcal{F}_{\mathcal{P}}$ are included in $\mathfrak{h}_0 \cap \mathcal{O}$. □

We need below the concept of a permutation matrix. Let $\mathfrak{S}_n$ be the symmetric group of degree $n$, which consists of all the permutations of $n$ letters. For $\pi \in \mathfrak{S}_n$, the $n \times n$ permutation matrix $P_\pi$ is defined to have rows equal to $\mathbf{e}_{\pi(1)}, \ldots, \mathbf{e}_{\pi(n)}$, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the standard basis of $\mathbb{R}^n$, i.e., the entries of $P_\pi$ are $(P_\pi)_{ij} = \delta_{\pi(i)j}$. Since $P_{\pi_1 \circ \pi_2} = P_{\pi_2} P_{\pi_1}$ for any $\pi_1, \pi_2 \in \mathfrak{S}_n$, we have $P_\pi^{-1} = P_{\pi^{-1}}$ and hence $\left(P_\pi^{-1}\right)_{ij} = (P_{\pi^{-1}})_{ij} = \delta_{\pi^{-1}(i)j} = \delta_{\pi(j)i} = (P_\pi)_{ji} = \left(P_\pi^T\right)_{ij}$, which shows that $P_\pi \in U(n)$.

The generic adjoint orbit $\mathcal{O}$ through $\sqrt{-1}\mathrm{diag}\,(x_1, \ldots, x_n)$ consists of all the invertible skew-Hermitian matrices whose eigenvalues are $\sqrt{-1}x_1, \ldots, \sqrt{-1}x_n$ (all $x_i$ are distinct and $x_i \neq 0$). Since $\mathfrak{h}_0$ is the set of all diagonal matrices whose entries are purely imaginary, it is obvious that

$$\mathfrak{h}_0 \cap \mathcal{O} = \left\{\sqrt{-1}\mathrm{diag}\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right) \mid \pi \in \mathfrak{S}_n\right\}.$$

However, since $\left(P_\pi^{-1}\right)_{ij} = \delta_{\pi(j)i}$, we have

$$\left(P_\pi \mathrm{diag}\,(x_1, \ldots, x_n)\, P_\pi^{-1}\right)_{ij} = \sum_{k=1}^n \delta_{\pi(i)k} x_k \delta_{\pi(j)k} = x_{\pi(i)} \delta_{\pi(i)\pi(j)} = x_{\pi(i)} \delta_{ij},$$

which shows that

$$\mathfrak{h}_0 \cap \mathcal{O} = \left\{\sqrt{-1} P_\pi \mathrm{diag}\,(x_1, \ldots, x_n)\, P_\pi^{-1} \,\middle|\, \pi \in \mathfrak{S}_n\right\}.$$

For a generic adjoint orbit $\mathcal{O}$, the set $\mathfrak{h}_0 \cap \mathcal{O}$ is clearly discrete and hence consists of the isolated equilibria for Euler equation (2.4) restricted to $\mathcal{O}$. Since all the isolated equilibria are common equilibria, $\mathfrak{h}_0 \cap \mathcal{O}$ is the set of isolated equilibria of the restriction of Euler equation to the generic orbit $\mathcal{O}$.

Next, we consider the non-degeneracy of the common equilibrium points of $\mathcal{F}_{\mathcal{P}}$. We begin with some general remarks.

**Definition 4.2.** *Let $(N, \Theta)$ be a $2n$-dimensional symplectic manifold and $f \in \mathcal{C}^\infty(N)$. Assume that the Hamiltonian system $\Xi_f$ is completely integrable whose Poisson commuting functionally independent first integrals are $f_1, \ldots, f_{n-1}, f_n = f$. The common equilibrium point $x_0 \in N$ for this set of first integrals, i.e., $\mathsf{d}f_1(x_0) = \cdots = \mathsf{d}f_n(x_0) = 0$, is* non-degenerate*, if the linearization of the Hamiltonian vector fields $\Xi_{f_1}, \ldots, \Xi_{f_n}$ at $x_0$ generate a Cartan subalgebra in the symplectic Lie algebra $\mathfrak{sp}\,(T_{x_0}N, \Theta_{x_0})$.*

One of the advantages of the non-degeneracy of the isolated equilibrium is the convergence of the Birkhoff normal forms defined around the equilibrium. In fact, for a non-degenerate isolated equilibrium of an analytic completely integrable system, Vey [35] has shown that the Birkhoff normal form of the Hamiltonian can be obtained via a convergent canonical transformation. (See [12] for the $\mathcal{C}^\infty$ case.) More precisely, given an isolated non-degenerate equilibrium $x_0$ for a completely integrable Hamiltonian system associated to the Hamiltonian $f$ on $2n$-dimensional phase space, we can take Darboux coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ such that the Hamiltonian $f$ is put into Birkhoff normal form $f = f\,(I_1, \ldots, I_n)$, where $I_1, \ldots, I_n$ are functionally independent quadratic functions in the coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ whose expressions are dictated by the Williamson normal form of the linearization of $\Xi_f(x_0)$. Now, assume that the linearization of $\Xi_f$ at $x_0 \in N$ is Lyapunov stable at the origin in $T_{x_0}N$, i.e., $x_0$ is linearly stable. By Vey's result, there are Darboux coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ around $x_0$ which put the Hamiltonian in Birkhoff normal form $f = f\left(\frac{q_1^2+p_1^2}{2}, \cdots, \frac{q_n^2+p_n^2}{2}\right)$. In these Darboux coordinates, the $n$ functions $\frac{q_1^2+p_1^2}{2}, \cdots, \frac{q_n^2+p_n^2}{2}$ are Poisson commuting first integrals for the Hamiltonian $f$. Consider the polydisc neighborhood

$$D(\epsilon) = \left\{ (q_1, ..., q_n, p_1, ..., p_n) \;\middle|\; \frac{p_1^2+q_1^2}{2} < \epsilon, ..., \frac{p_n^2+q_n^2}{2} < \epsilon \right\},$$

for sufficiently small $\epsilon > 0$. For any $(0 \leq)\delta < \epsilon$, the Hamiltonian vector field $\Xi_f$ can be restricted to the intersection of the level hypersurfaces

$$\partial D(\delta) = \left\{ (p_1, ..., p_n; q_1, ..., q_n) \;\middle|\; \frac{p_1^2+q_1^2}{2} = \delta, ..., \frac{p_n^2+q_n^2}{2} = \delta \right\}.$$

Now, we apply the existence theorem for long time solutions of ordinary differential equations on compact manifolds [2, Corollary 2, §35, Chapter 5, page 305] to the restriction $\Xi_f|_{\partial D(\delta)}$. Since $\partial D(\delta)$ is a compact manifold invariant under the flow of $\Xi_f$, all the integral curves of $\Xi_f|_{\partial D(\delta)}$ exist for all time. Therefore, any integral curve of $\Xi_f$ which starts at point in $D(\epsilon)$ exists for all time and is included in $D(\epsilon)$. Therefore, the equilibrium $x_0$ is Lyapunov (nonlinearly) stable. To sum up, we have the following proposition.

**Proposition 4.3.** *Let $\Xi_f$ be a completely integrable Hamiltonian vector field. Then, a non-degenerate isolated linearly stable equilibrium is Lyapunov (nonlinearly) stable.*

A criterion guaranteeing the non-degeneracy of a common equilibrium point is given by the following result.

**Theorem 4.4** (Bolsinov-Oshemkov [8])**.** *Assume that the conditions in Theorem 4.1 hold. Let $x \in M$ be a common equilibrium point. Suppose that the rank of a Poisson bracket $\mathcal{A}_\lambda(x)$ does not attain the maximum exactly when $\lambda = \lambda_1, \ldots, \lambda_q$, $q = \frac{1}{2}(\dim M - \operatorname{corank}\mathcal{A})$, where $\lambda_1, \ldots, \lambda_q$ are distinct, and that there exists $f \in \mathcal{F}_\mathcal{P}$ for which the linearization of the Hamiltonian vector field $\Xi_f$ at $x$ is non-degenerate as a linear endomorphism of the tangent space $T_x\mathcal{O}$ of the symplectic leaf $\mathcal{O}$ containing $x$. Then $x$ is non-degenerate.*

We apply this theorem to the common equilibrium points for the $U(n)$ free rigid body. Take $X = \sqrt{-1}\operatorname{diag}(x_1, \ldots, x_n) \in \mathfrak{h}_0$, where $x_j \in \mathbb{R}$, $j = 1, \ldots, n$, are distinct. We first need to find those $\lambda$'s for which the Poisson bracket $\{\cdot,\cdot\}_{\mathsf{E}+\lambda\mathsf{J}^2}$ does not attain the maximum rank.

**(a)** Assume first that $\mathsf{A} := \mathsf{E} + \lambda\mathsf{J}^2$ is non-degenerate. We use the isomorphism $\Psi_\mathsf{A} : \mathfrak{u}(n)_\mathsf{A} \to \mathfrak{u}(p,q)$ (cf. Proposition 2.5), where $(p,q)$ is the signature of the matrix $\mathsf{A}$, to describe the Poisson bracket $\{\cdot,\cdot\}_\mathsf{A}$. On $\mathfrak{u}(p,q)$, let $\langle\cdot,\cdot\rangle_{\mathfrak{u}(p,q)}$ denote the invariant non-degenerate bilinear form, defined by

$$\langle Z, W\rangle_{\mathfrak{u}(p,q)} = -\operatorname{Tr}(ZW) = \operatorname{Tr}(\mathsf{E}_{p,q}Z^*\mathsf{E}_{p,q}W), \quad Z, W \in \mathfrak{u}(p,q).$$

Note that, in general, there is a difference between ${\Psi_\mathsf{A}}^{-1} : \mathfrak{u}(p,q) \to S$ and $\Psi_{\mathsf{A}^{-1}} : S \to \mathfrak{u}(p,q)$, where $S$ is the underlying vector space underlying both Lie algebras $\mathfrak{u}(n)_\mathsf{A}$ and $\mathfrak{u}(n)_{\mathsf{A}^{-1}}$. With these conventions, for $X \in S$ and $Y \in \mathfrak{u}(p,q)$, we have:

$$\begin{aligned}
\left\langle X, {\Psi_{\mathsf{A}}}^{-1}(Y)\right\rangle &= -\mathrm{Tr}\left(X{\Psi_{\mathsf{A}}}^{-1}(Y)\right)\\
&= -\mathrm{Tr}\left(X\sqrt{\mathsf{A}}^{-1}Y\sqrt{\mathsf{A}}^{-1}\right)\\
&= -\mathrm{Tr}\left(X\sqrt{\mathsf{A}^{-1}}Y\sqrt{\mathsf{A}^{-1}}\right)\\
&= -\mathrm{Tr}\left(\sqrt{\mathsf{A}^{-1}}X\sqrt{\mathsf{A}^{-1}}Y\right)\\
&= -\mathrm{Tr}\left(\Psi_{\mathsf{A}^{-1}}(X)Y\right)\\
&= \langle \Psi_{\mathsf{A}^{-1}}(X), Y\rangle_{\mathfrak{u}(p,q)}.
\end{aligned}$$

For smooth $F, G \in \mathcal{C}^\infty\left(\mathfrak{u}(n)_{\mathsf{A}}{}^*\right)$, we have, at $X \in S$ (the vector space of all skew-Hermitian matrices underlying $\mathfrak{u}(n)_{\mathsf{A}}{}^*$),

$$\begin{aligned}
\{F,G\}_{\mathsf{A}}(X) &= \langle X, [\nabla F(X), \nabla G(X)]_{\mathsf{A}}\rangle\\
&= \left\langle X, {\Psi_{\mathsf{A}}}^{-1}\left([\Psi_{\mathsf{A}}(\nabla F(X)), \Psi_{\mathsf{A}}(\nabla G(X))]\right)\right\rangle\\
&= \langle \Psi_{\mathsf{A}^{-1}}(X), [\Psi_{\mathsf{A}}(\nabla F(X)), \Psi_{\mathsf{A}}(\nabla G(X))]\rangle_{\mathfrak{u}(p,q)}.
\end{aligned}$$

By the above description of the Poisson bracket $\{\cdot,\cdot\}_{\mathsf{A}}$, we see that it does not attain the maximum rank if and only if the element $\Psi_{\mathsf{A}^{-1}}(X) \in \mathfrak{u}(p,q)$ is singular in the sense that its eigenvalues are not distinct. If $X$ is a common equilibrium point, we can write it as $X = \sqrt{-1}\mathrm{diag}\,(x_1,\ldots,x_n)$ and we obviously have $\Psi_{\mathsf{A}^{-1}}(X) = \sqrt{-1}\mathrm{diag}\left(\frac{x_1}{a_1},\ldots,\frac{x_n}{a_n}\right)$, where $\mathsf{A} = \mathsf{E} + \lambda\mathsf{J}^2 = \mathrm{diag}\,(a_1,\ldots,a_n)$, i.e. $a_j = 1+\lambda J_j^2$, $j = 1,\ldots,n$. Thus, $\Psi_{\mathsf{A}^{-1}}(X)$ is not singular if and only if $\frac{x_1}{a_1} = \frac{x_1}{1+\lambda J_1^2}, \ldots, \frac{x_n}{a_n} = \frac{x_n}{1+\lambda J_n^2}$ are distinct, which is equivalent to $\lambda \neq \frac{x_j - x_i}{x_i J_j^2 - x_j J_i^2}$, for all $1 \le i < j \le n$.

**(b)** If $\mathsf{A} = \mathsf{E} + \lambda\mathsf{J}^2 = \mathrm{diag}(a_1,\ldots,a_n)$ is degenerate and $\lambda \neq 0$ (if $\lambda = 0$ we are in case (a)), its rank is $n-1$ because all entries of $\mathsf{J}$ are distinct. Without loss of generality, assume that $a_n = 0$, i.e., $\lambda = -\frac{1}{J_n^2}$. In this case, we use the isomorphism $\Psi'_{\mathsf{A}} : \mathfrak{u}(n)_{\mathsf{A}} \to \mathfrak{u}(p',q')_{\mathsf{E}'}$ instead of $\Psi_{\mathsf{A}}$; see Lemma 3.13 of Subsection 3.3.

On $\mathfrak{u}(p',q')_{\mathsf{E}'}$, let $\langle\cdot,\cdot\rangle_{\mathfrak{u}(p',q')_{\mathsf{E}'}}$ denote the non-degenerate bilinear form, defined by

$$\langle Z, W\rangle_{\mathfrak{u}(p',q')_{\mathsf{E}'}} = -\mathrm{Tr}\,(ZW) = \mathrm{Tr}\left(\mathsf{E}_{p',q'+1}Z^*\mathsf{E}_{p',q'+1}W\right), \quad Z, W \in \mathfrak{u}(p',q')_{\mathsf{E}'}.$$

Note that this bilinear form is not invariant under the Lie bracket $[\cdot,\cdot]_{\mathsf{E}'}$ of $\mathfrak{u}(p',q')_{\mathsf{E}'}$. Let $\mathsf{A}' = \mathrm{diag}\,(a_1,\ldots,a_{n-1})$ and $\mathsf{A}' \oplus 1 := \mathrm{diag}\,(a_1,\ldots,a_{n-1},1)$. For $X \in S$ and $Y \in \mathfrak{u}(p',q')_{\mathsf{E}'}$, we have

$$\begin{aligned}
\left\langle X, {\Psi'_{\mathsf{A}}}^{-1}(Y)\right\rangle &= -\mathrm{Tr}\left(X{\Psi'_{\mathsf{A}}}^{-1}(Y)\right)\\
&= -\mathrm{Tr}\left(X\sqrt{\mathsf{A}'\oplus 1}^{-1}Y\sqrt{\mathsf{A}'\oplus 1}^{-1}\right)\\
&= -\mathrm{Tr}\left(X\sqrt{\mathsf{A}'^{-1}\oplus 1}Y\sqrt{\mathsf{A}'^{-1}\oplus 1}\right)\\
&= -\mathrm{Tr}\left(\sqrt{\mathsf{A}'^{-1}\oplus 1}X\sqrt{\mathsf{A}'^{-1}\oplus 1}Y\right)\\
&= -\mathrm{Tr}\left(\Psi'_{\mathsf{A}'^{-1}\oplus 1}(X)Y\right)\\
&= \left\langle \Psi'_{\mathsf{A}'^{-1}\oplus 1}(X), Y\right\rangle_{\mathfrak{u}(p',q')_{\mathsf{E}'}}.
\end{aligned}$$

The Poisson bracket $\{\cdot,\cdot\}_{\mathsf{A}}$ can be expressed at $X \in S$ (the vector space of all skew-Hermitian matrices underlying $\mathfrak{u}(n)_{\mathsf{A}}{}^*$) as

$$\begin{aligned}\{F,G\}_{\mathsf{A}}(X) &= \langle X, [\nabla F(X), \nabla G(X)]_{\mathsf{A}}\rangle \\ &= \left\langle X, \Psi_{\mathsf{A}}'^{-1}\left([\Psi_{\mathsf{A}}'(\nabla F(X)), \Psi_{\mathsf{A}}'(\nabla G(X))]_{\mathsf{E}'}\right)\right\rangle \\ &= \left\langle \Psi_{\mathsf{A}'^{-1}\oplus 1}'(X), [\Psi_{\mathsf{A}}'(\nabla F(X)), \Psi_{\mathsf{A}}'(\nabla G(X))]_{\mathsf{E}'}\right\rangle_{\mathfrak{u}(p',q')_{\mathsf{E}'}}\end{aligned}$$

for smooth functions $F$ and $G$ on $\mathfrak{u}(n)_{\mathsf{A}}{}^*$.

Recall that $\left(\Psi_{\mathsf{A}'^{-1}\oplus 1}'(X)\right) = \sqrt{-1}\mathrm{diag}\left(\frac{x_1}{a_1}, \ldots, \frac{x_{n-1}}{a_{n-1}}, x_n\right)$, for $X = \sqrt{-1}\mathrm{diag}(x_1, \ldots, x_n)$. Putting $\Psi_{\mathsf{A}}'(\nabla F(X)) =: Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22}\end{bmatrix}$, $\Psi_{\mathsf{A}}'(\nabla G(X)) =: Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22}\end{bmatrix}$, where $Y_{11}, Z_{11} \in \mathfrak{u}(p',q')$, $p'+q' = n-1$, $Y_{12}, Z_{12} \in \mathbb{C}^{(n-1)\times 1}$, $Y_{21} = Y_{12}^*\mathsf{E}_{p',q'}, Z_{21} = Z_{12}^*\mathsf{E}_{p',q'} \in \mathbb{C}^{1\times(n-1)}$, $Y_{22}, Z_{22} \in \sqrt{-1}\mathbb{R}$, we calculate $\{F,G\}_{\mathsf{A}}(X)$ as follows:

$$\begin{aligned}\{F,G\}_{\mathsf{A}}(X) &= \left\langle \Psi_{\mathsf{A}'^{-1}\oplus 1}'(X), [Y,Z]_{\mathsf{E}'}\right\rangle_{\mathfrak{u}(p',q')_{\mathsf{E}'}} \\ &= \left\langle \sqrt{-1}\mathrm{diag}\left(\frac{x_1}{a_1}, \ldots, \frac{x_{n-1}}{a_{n-1}}, x_n\right), \begin{bmatrix} Y_{11}Z_{11} - Z_{11}Y_{11} & Y_{11}Z_{12} - Z_{11}Y_{12} \\ Y_{21}Z_{11} - Z_{21}Y_{11} & Y_{21}Z_{12} - Z_{21}Y_{12}\end{bmatrix}\right\rangle_{\mathfrak{u}(p',q')_{\mathsf{E}'}} \\ &= -\sqrt{-1}\left\{\mathrm{Tr}\left(\mathrm{diag}\left(\frac{x_1}{a_1}, \ldots, \frac{x_{n-1}}{a_{n-1}}\right)(Y_{11}Z_{11} - Z_{11}Y_{11})\right) + x_n(Y_{21}Z_{12} - Z_{21}Y_{12})\right\}.\end{aligned}$$

Writing $Y_{11} = (y_{ij})$, $Z_{11} = (z_{ij})$, $i,j = 1, \ldots, n-1$, we have

$$\{F,G\}_{\mathsf{A}}(X) = -\sqrt{-1}\left\{\sum_{i,j=1}^{n-1}\frac{x_i}{a_i}(y_{ij}z_{ji} - z_{ij}y_{ji}) + x_n(Y_{21}Z_{12} - Z_{21}Y_{12})\right\},$$

which has real maximal rank $n^2 - n$ as a bilinear form in $(Y,Z)$ if and only if $\frac{x_j}{a_j} = \frac{x_j}{1+\lambda J_j^2}$, $j = 1, \ldots, n-1$, are distinct, which is equivalent to $\lambda \neq \frac{x_j - x_i}{x_i J_j^2 - x_j J_i^2}$, $1 \le i < j \le n-1$, and $x_n \neq 0$. Recall that a point $X \in \mathfrak{u}(n)$ is called generic if the eigenvalues of $X$ are distinct and $X$ is invertible. Thus, the condition $x_n \neq 0$ is satisfied if $X$ is generic. Since we have assumed $\lambda = -\frac{1}{J_n^2}$, we see that $\mathrm{rank}\{\cdot,\cdot\}_{\mathsf{A}}(X) = n^2 - n$, if $-\frac{1}{J_n^2}$ is distinct from $\frac{x_j - x_i}{x_i J_j^2 - x_j J_i^2}$ for any $1 \le i < j \le n-1$, which is satisfied for generic $X \in \mathfrak{h}_0$.

To sum up the arguments (a) and (b), we conclude that the rank of $\{\cdot,\cdot\}_{\mathsf{E}+\lambda\mathsf{J}^2}(X)$ drops from the maximal rank $n^2 - n$ exactly when $\lambda = \frac{x_j - x_i}{x_i J_j^2 - x_j J_i^2}$ for some $1 \le i < j \le n$, when the orbit $\mathcal{O}$ is generic in the sense that it consists of invertible matrices with distinct eigenvalues satisfying $\frac{x_j - x_i}{x_i J_j^2 - x_j J_i^2} \neq -\frac{1}{J_k^2}$ for any $1 \le i < j \le n$ and $k = 1, \ldots, n$, where $x_1, \ldots, x_n$ are the eigenvalues of the matrices in the orbit $\mathcal{O}$. The latter genericity condition is equivalent to $\frac{x_i}{x_j} \neq \frac{J_i^2 - J_k^2}{J_j^2 - J_k^2}$ for all $i,j,k = 1, \ldots, n$. Thus, if $\frac{x_j - x_i}{x_i J_j^2 - x_j J_i^2}$, $1 \le i < j \le n$, are distinct, the number of the parameters $\lambda$ for which the rank of $\{\cdot,\cdot\}_{\mathsf{E}+\lambda\mathsf{J}^2}(X)$ at $X \in \mathfrak{h}_0 \cap \mathcal{O}$ drops from the maximal rank

$n^2-n$ is $\frac{1}{2}(n^2-n)=\frac{1}{2}\left(\dim\left(\mathfrak{u}(n)\right)-\operatorname{corank}\left(\{\cdot,\cdot\}\right)\right)$ for such generic orbits $\mathcal{O}$. By Theorem 4.4, we have the following result.

**Theorem 4.5.** *The common equilibrium points $X=\sqrt{-1}\mathrm{diag}\left(x_1,\dots,x_n\right)\in\mathfrak{h}_0\cap\mathcal{O}$ of the $U(n)$ free rigid body, where $\mathcal{O}$ is a generic orbit consisting of invertible matrices with distinct eigenvalues, is non-degenerate if all the numbers in the set*

$$\left\{\frac{x_j-x_i}{x_iJ_j^2-x_jJ_i^2},\,-\frac{1}{J_k^2}\;\middle|\;1\le i<j\le n,\,k=1,\dots,n\right\}$$

*are distinct.*

*Proof.* By Theorem 4.4, all that remains to be shown is that there is a function in $\mathcal{F}_{\mathcal{P}}$, for which the linearized Hamiltonian vector field is non-degenerate. We take the Manakov integral $I_2^{(3)}(X)=3\mathrm{Tr}\left(\mathsf{J}^2X^2\right)$, whose gradient vector field is $\nabla I_2^{(3)}(X)=-3\left(X\mathsf{J}^2+\mathsf{J}^2X\right)$. So, its Hamiltonian vector field $\Xi_{I_2^{(3)}}$ with respect to the Lie-Poisson bracket $\{\cdot,\cdot\}$ is

$$\Xi_{I_2^{(3)}}(X)=\left[X,-3\left(X\mathsf{J}^2+\mathsf{J}^2X\right)\right]=-3\left[X^2,\mathsf{J}^2\right].$$

The linearization of this Hamiltonian vector field at the common equilibrium points $X\in\mathfrak{h}_0\cap\mathcal{O}$ is the endomorphism of $T_X\mathcal{O}$ given for $Y\in T_X\mathcal{O}$ by

$$Y\mapsto-3\left[YX+XY,\mathsf{J}^2\right].$$

Note that the tangent space $T_X\mathcal{O}\subset\mathfrak{u}(n)$ is the orthogonal complement of $\mathfrak{h}_0$ with respect to the inner product $\langle\cdot,\cdot\rangle$. The kernel of this linear endomorphism is $\{Y\in T_X\mathcal{O}\mid YX+XY \text{ is diagonal}\}$. Assume that $Y=(y_{ij})$, $i,j=1,\dots,n$, is in this kernel. Since $X=\sqrt{-1}\mathrm{diag}\left(x_1,\dots,x_n\right)$, we have

$$YX+XY=\sqrt{-1}\left((x_i+x_j)y_{ij}\right).$$

This matrix is diagonal if and only if $Y$ is diagonal, i.e. $Y\in\mathfrak{h}_0$, under the condition $x_i+x_j\neq0$ for all $1\le i<j\le n$, which is satisfied for generic $X\in\mathfrak{u}(n)$. This means $Y=0$ for generic adjoint orbits $\mathcal{O}$, since $T_X\mathcal{O}\cap\mathfrak{h}_0=\{0\}$. □

## 5 Lyapunov stability of equilibria

In this section, we study the nonlinear stability for the equilibria of the $U(n)$ free rigid body dynamics. As we have seen above, the Euler equation for this dynamical system induces a Hamiltonian system on the (co)adjoint orbits $\mathcal{O}$, which can also be understood as the reduced system of the Hamiltonian system on $T^*U(n)$ by Marsden-Weinstein reduction. Hence we investigate the stability of the equilibria in $\mathfrak{h}_0\cap\mathcal{O}$ of the Euler equation (2.4) on a generic orbit $\mathcal{O}\subset\mathfrak{u}(n)$. To do this, we analyze the linear stability of each equilibrium in $\mathfrak{h}_0\cap\mathcal{O}$, i.e., the Lyapunov stability of the origin of the linearized system at such an equilibrium. As we shall see, all such equilibria are linearly stable. The non-degeneracy condition of these equilibria implies then their nonlinear stability using the result of Vey [35]. See Proposition 4.3.

We start with a general description of the linearization of a Hamiltonian vector field $\Xi_H$ for the Hamiltonian $H$ on a symplectic manifold $(M,\omega)$ at a critical point $x_0\in M$, i.e., $\Xi_H(x_0)=0$. Let $X$ and $Y$ be arbitrary non-vanishing vector fields defined in a neighborhood of $x_0$. Let $\phi_X^t$ be the flow of $X$ and denote by $\mathcal{L}_X$ the Lie derivative in the direction $X$. Taking $\left.\frac{\mathsf{d}}{\mathsf{d}t}\right|_{t=0}$ of both sides of the identity $(\phi_X^t)^*\left(\omega(\Xi_H, Y)\right)(x_0) = (\phi_X^t)^*\left(\mathsf{d}H \cdot Y\right)(x_0)$ and using $\Xi_H(x_0) = 0$, $\mathsf{d}H(x_0) = 0$, yields

$$\omega(x_0)\left([X, \Xi_H](x_0), Y(x_0)\right) = \left(\mathsf{d}\mathcal{L}_X H\right)(x_0) \cdot Y. \tag{5.1}$$

Writing in coordinates

$$\Xi_H = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}, \quad X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}, \quad Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial x^i}, \quad \omega = \sum_{i,j=1}^n \omega_{ij} \mathsf{d}x^i \wedge \mathsf{d}x^j,$$

we have $v^i(x_0) = 0$ and $\dfrac{\partial H}{\partial x^j}(x_0) = 0$ and hence

$$[X, \Xi_H](x_0) = \sum_{i,j=1}^n X^j(x_0) \frac{\partial v^i}{\partial x^j}(x_0) \frac{\partial}{\partial x^i}, \qquad \left(\mathsf{d}\mathcal{L}_X H\right)(x_0) = \sum_{i,j=1}^n X^i(x_0) \frac{\partial^2 H}{\partial x^i \partial x^j}(x_0) \mathsf{d}x^j.$$

Formula (5.1) can hence be written as

$$\sum_{i,j,k=1}^n \omega_{ij}(x_0) \frac{\partial v^i}{\partial x^k}(x_0) X^k(x_0) Y^j(x_0) = \sum_{i,j=1}^n Y^j(x_0) X^i(x_0) \frac{\partial^2 H}{\partial x^i \partial x^j}(x_0). \tag{5.2}$$

Since both $X(x_0)$ and $Y(x_0)$ are arbitrary, denoting the inverse of the matrix $(\omega_{ij}(x_0))$ by $\left(\omega^{ij}(x_0)\right)$, this identity implies

$$\frac{\partial v^i}{\partial x^j}(x_0) = \sum_{k=1}^n \omega^{ik}(x_0) \frac{\partial^2 H}{\partial x^k \partial x^j}(x_0), \tag{5.3}$$

which proves the following result.

**Proposition 5.1.** *Let $x_0 \in M$ be an equilibrium of the Hamiltonian system $\Xi_H \in \mathfrak{X}(M)$. The linearization at $x_0$ of $\Xi_H$ is given by the linear differential equation for $P(t) \in T_{x_0}M$*

$$\frac{\mathsf{d}}{\mathsf{d}t} P(t) = \omega(x_0)^{-1} \operatorname{Hess}(H)(x_0) P(t), \tag{5.4}$$

*where the symplectic form $\omega(x_0)$ and the Hessian $\operatorname{Hess}(H)(x_0)$ of the Hamiltonian $H$ at $x_0$ are regarded as linear endomorphisms of $T_{x_0}M$.*

*The Hamiltonian vector field $\Xi_H$ is linearly stable at $x_0$ if and only if the matrix $\omega(x_0)^{-1}\operatorname{Hess}(H)(x_0)$ has only purely imaginary eigenvalues.*

We apply these general results to the Euler equations (2.4) on a generic $U(n)$-(co)adjoint orbit $\mathcal{O}$. In order to calculate the matrix representations of the orbit symplectic form $\omega$ on $\mathcal{O}$ and the Hessian of the Hamiltonian $H|_{\mathcal{O}}$, we introduce the following matrices. Let $\mathsf{E}_{ij}$ be the matrix whose only nonzero entry 1 is in the $(i, j)$ component. The matrices $\mathsf{H}_k := \sqrt{-1}\mathsf{E}_{kk}$, $1 \leq k \leq n$, $\mathsf{X}_{ij} := \dfrac{1}{\sqrt{2}}(\mathsf{E}_{ij} - \mathsf{E}_{ji})$, $\mathsf{Y}_{ij} := \dfrac{\sqrt{-1}}{\sqrt{2}}(\mathsf{E}_{ij} + \mathsf{E}_{ji})$, $1 \leq i, j \leq n$, generate the Lie algebra $\mathfrak{u}(n)$. We have the following commutation relation between these generators:

$$\begin{aligned}
&[\mathsf{H}_i, \mathsf{H}_j] = 0, \quad 1 \leq i, j \leq n, \\
&[\mathsf{X}_{ij}, \mathsf{X}_{jk}] = \frac{1}{\sqrt{2}}\mathsf{X}_{ik}, \quad [\mathsf{Y}_{ij}, \mathsf{Y}_{jk}] = -\frac{1}{\sqrt{2}}\mathsf{Y}_{ik}, \quad [\mathsf{X}_{ij}, \mathsf{Y}_{jk}] = \frac{1}{\sqrt{2}}\mathsf{Y}_{ik}, \quad i < j < k, \\
&[\mathsf{X}_{ij}, \mathsf{Y}_{ij}] = \mathsf{H}_i - \mathsf{H}_j, \quad [\mathsf{H}_i, \mathsf{X}_{ij}] = \mathsf{Y}_{ij}, \quad [\mathsf{H}_i, \mathsf{Y}_{ij}] = -\mathsf{X}_{ij}, \quad i < j,
\end{aligned}$$

all other relations being zero or deduced easily from this list. The matrices $\mathsf{H}_1, \ldots, \mathsf{H}_n$ generate the standard Cartan subalgebra $\mathfrak{h}_0$ in $\mathfrak{u}(n)$ (the purely imaginary diagonal matrices). We have $\mathsf{E}_{ij} = \dfrac{1}{\sqrt{2}}\left(\mathsf{X}_{ij} - \sqrt{-1}\mathsf{Y}_{ij}\right)$ and $[\mathsf{H}_i - \mathsf{H}_j, \mathsf{E}_{ij}] = 2\sqrt{-1}\mathsf{E}_{ij} = \sqrt{2}\left(\sqrt{-1}\mathsf{X}_{ij} + \mathsf{Y}_{ij}\right)$. Choosing a basis $\eta_1, \ldots, \eta_n$ of $\sqrt{-1}\mathfrak{h}_0^*$ such that $\eta_i \cdot \mathsf{H}_j = \sqrt{-1}\delta_{ij}$, $1 \leq i < j \leq n$, the matrices $\mathsf{E}_{ij} \in \mathfrak{sl}(n, \mathbb{C}) \subset \mathfrak{u}(n) \otimes \mathbb{C} = \mathfrak{gl}(n, \mathbb{C})$, $1 \leq i < j \leq n$, are the root vectors of the complex simple Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ corresponding to the root $\eta_i - \eta_j$ and the matrices $\mathsf{X}_{ij}$ and $\mathsf{Y}_{ij}$ are its real and imaginary parts up to the multiple $\dfrac{1}{\sqrt{2}}$.

On the generic (co)adjoint orbit $\mathcal{O} \subset \mathfrak{u}(n)$, we take a common equilibrium point $X_0 \in \mathfrak{h}_0 \cap \mathcal{O}$ of the $U(n)$ free rigid body dynamics and express it as $X_0 = \displaystyle\sum_{i=1}^n x_i \mathsf{H}_i$, where $x_1, \ldots, x_n \in \mathbb{R}$ are distinct. Denoting the Cartan decomposition of $\mathfrak{u}(n)$ with respect to the standard Cartan subalgebra $\mathfrak{h}_0$ by $\mathfrak{u}(n) = \mathfrak{h}_0 \dotplus \mathfrak{m}_0$, where $\mathfrak{m}_0$ is the orthogonal complement with respect to the inner product $\langle X, Y \rangle := -\mathrm{Tr}\,(XY)$, for all $X, Y \in \mathfrak{u}(n)$, we see that $\mathfrak{m}_0$ can be identified with the tangent space $T_{X_0}\mathcal{O}$. Since $[\mathfrak{h}_0, \mathfrak{m}_0] = \mathfrak{m}_0$, we can take the following basis of $T_{X_0}\mathcal{O} \equiv \mathfrak{m}_0$:

$$\begin{aligned} \mathrm{ad}_{\mathsf{X}_{ij}} X_0 &= [\mathsf{X}_{ij}, X_0] = \sum_{k=1}^n x_k [\mathsf{X}_{ij}, \mathsf{H}_k] = -(x_i - x_j)\,\mathsf{Y}_{ij}, \\ \mathrm{ad}_{\mathsf{Y}_{ij}} X_0 &= [\mathsf{Y}_{ij}, X_0] = \sum_{k=1}^n x_k [\mathsf{Y}_{ij}, \mathsf{H}_k] = (x_i - x_j)\,\mathsf{X}_{ij}, \end{aligned}$$

where $1 \leq i < j \leq n$.

The orbit symplectic form $\omega$ on $\mathcal{O}$ is defined at a point $X \in \mathcal{O}$ by

$$\omega(X)\left(\mathrm{ad}_A X, \mathrm{ad}_B X\right) = \langle X, [A, B] \rangle,$$

where $\mathrm{ad}_A X$ and $\mathrm{ad}_B X$ are regarded as elements in $T_X\mathcal{O}$ and $A, B \in \mathfrak{u}(n)$. Since

$$\omega(X_0)\left(\mathrm{ad}_{X_{ij}} X_0, \mathrm{ad}_{Y_{ij}} X_0\right) = \langle X_0, [\mathsf{X}_{ij}, \mathsf{Y}_{ij}] \rangle = \left\langle \sum_{k-1}^n x_k \mathsf{H}_k, \mathsf{H}_i - \mathsf{H}_j \right\rangle = x_i - x_j,$$

it follows that the matrix representation of $\omega(X_0)$ with respect to the basis elements $\mathrm{ad}_{X_{ij}} X_0$ and $\mathrm{ad}_{Y_{ij}} X_0$, $1 \leq i < j \leq n$, is

$$\begin{bmatrix} \omega(X_0)\left(\mathrm{ad}_{X_{ij}} X_0, \mathrm{ad}_{X_{ij}} X_0\right) & \omega(X_0)\left(\mathrm{ad}_{X_{ij}} X_0, \mathrm{ad}_{Y_{ij}} X_0\right) \\ \omega(X_0)\left(\mathrm{ad}_{Y_{ij}} X_0, \mathrm{ad}_{X_{ij}} X_0\right) & \omega(X_0)\left(\mathrm{ad}_{Y_{ij}} X_0, \mathrm{ad}_{Y_{ij}} X_0\right) \end{bmatrix} = (x_i - x_j) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The other components of the matrix representation of $\omega(X_0)$ are zero. These considerations give the following result.

**Proposition 5.2.** *The orbit symplectic form $\omega(X_0)$ at $X_0 = \displaystyle\sum_{k=1}^n x_k \mathsf{H}_k$ can be represented by the direct sum of $2 \times 2$ blocks as*

$$\bigoplus_{1 \leq i < j \leq n} \left( (x_i - x_j) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right),$$

*with respect to the basis $\mathrm{ad}_{\mathsf{X}_{ij}} X_0$ and $\mathrm{ad}_{\mathsf{Y}_{ij}} X_0$, $1 \leq i < j \leq n$, of $T_{X_0}\mathcal{O}$. Here, the direct sum is that of linear endomorphisms of $\displaystyle\bigoplus_{1 \leq i < j \leq n} \mathrm{span}\left\{\mathrm{ad}_{\mathsf{X}_{ij}} X_0, \mathrm{ad}_{\mathsf{Y}_{ij}} X_0\right\} = T^*_{X_0}\mathcal{O}$.*

We calculate the Hessian of the Hamiltonian $H|_{\mathcal{O}}$ restricted to the orbit $\mathcal{O}$. To this end, we describe the Hessian of a general function $f \in \mathcal{C}^\infty\left(\mathfrak{u}(n)^*\right)$ restricted to $\mathcal{O}$. Take again a point $X_0 = \sum_{i=1}^{n} x_i \mathsf{H}_i \in \mathfrak{h}_0 \cap \mathcal{O}$. The gradient of $f|_{\mathcal{O}}$ at $X_0$ with respect to the induced Riemannian metric from $\langle \cdot, \cdot \rangle$ is given as

$$\mathrm{grad} f|_{\mathcal{O}}(X_0) = \nabla f(X_0) - \sum_{i=1}^{n} \langle \nabla f(X_0), \mathsf{H}_i \rangle \, \mathsf{H}_i,$$

since $\mathsf{H}_1, \dots, \mathsf{H}_n$ form an orthonormal basis of $\mathfrak{h}_0 \equiv T_{X_0}\mathcal{O}^\perp$ with respect to $\langle \cdot, \cdot \rangle$. For an arbitrary element $g \in U(n)$, we can give an expression of the gradient of $f|_{\mathcal{O}}$ at $\mathrm{Ad}_g X_0$ as

$$\mathrm{grad} f|_{\mathcal{O}}(\mathrm{Ad}_g X_0) = \nabla f(\mathrm{Ad}_g X_0) - \sum_{i=1}^{n} \langle \nabla f(\mathrm{Ad}_g X_0), \mathrm{Ad}_g \mathsf{H}_i \rangle \, \mathrm{Ad}_g \mathsf{H}_i.$$

Clearly, $\mathrm{Ad}_g\mathsf{H}_1, \dots, \mathrm{Ad}_g\mathsf{H}_n$ form an orthonormal basis of $\left(T_{\mathrm{Ad}_g X_0}\mathcal{O}\right)^\perp \equiv \mathrm{Ad}_g\mathfrak{h}_0$. Taking arbitrary elements $A, B \in \mathfrak{u}(n)$, we calculate the Hessian of $f|_{\mathcal{O}}$ at $X_0$ as a quadratic form in $\mathrm{ad}_A X_0, \mathrm{ad}_B X_0 \in T_{X_0}\mathcal{O}$ as follows: Since

$$\begin{aligned}
\left.\frac{\mathsf{d}}{\mathsf{d}t}\right|_{t=0} \mathrm{grad} f|_{\mathcal{O}}\left(\mathrm{Ad}_{e^{tB}} X_0\right) = \left.\frac{\mathsf{d}}{\mathsf{d}t}\right|_{t=0} \nabla f(\mathrm{Ad}_{e^{tB}} X_0) &- \sum_{i=1}^{n} \left\langle \left.\frac{\mathsf{d}}{\mathsf{d}t}\right|_{t=0} \nabla f\left(\mathrm{Ad}_{e^{tB}} X_0\right), \mathsf{H}_i \right\rangle \mathsf{H}_i \\
&- \sum_{i=1}^{n} \left\langle \nabla f(X_0), \left.\frac{\mathsf{d}}{\mathsf{d}t}\right|_{t=0} \mathrm{Ad}_{e^{tB}} \mathsf{H}_i \right\rangle \mathsf{H}_i \\
&- \sum_{i=1}^{n} \langle \nabla f(X_0), \mathsf{H}_i \rangle \left.\frac{\mathsf{d}}{\mathsf{d}t}\right|_{t=0} \mathrm{Ad}_{e^{tB}} \mathsf{H}_i,
\end{aligned}$$

we have

$$\begin{aligned}
\langle \mathrm{ad}_A X_0, \mathrm{Hess} f|_{\mathcal{O}}(X_0) \cdot \mathrm{ad}_B X_0 \rangle &= \left\langle \mathrm{ad}_A X_0, \left.\frac{\mathsf{d}}{\mathsf{d}t}\right|_{t=0} \mathrm{grad} f|_{\mathcal{O}}\left(\mathrm{Ad}_{e^{tB}} X_0\right) \right\rangle \\
&= \left\langle \mathrm{ad}_A X_0, \left.\frac{\mathsf{d}}{\mathsf{d}t}\right|_{t=0} \nabla f\left(\mathrm{Ad}_{e^{tB}} X_0\right) \right\rangle \\
&\quad - \sum_{i=1}^{n} \langle \nabla f(X_0), \mathsf{H}_i \rangle \cdot \langle \mathrm{ad}_A X_0, \mathrm{ad}_B \mathsf{H}_i \rangle \, ;
\end{aligned}$$

note that the second and third summand vanish because $\mathrm{ad}_{X_0} A \in T_{X_0}\mathcal{O}$ and $\mathsf{H}_i \in \mathfrak{h}_0 = (T_{X_0}\mathcal{O})^\perp$.

We use this formula to calculate the Hessian of the Hamiltonian $H|_{\mathcal{O}}$ restricted to the orbit $\mathcal{O}$ as a quadratic form over $T_{X_0}\mathcal{O}$ with respect to the basis $\mathrm{ad}_{X_{ij}} X_0$, $\mathrm{ad}_{Y_{ij}} X_0$, $1 \le i < j \le n$. For the Hamiltonian $H(X) = \frac{1}{2}\langle X, \mathcal{J}^{-1}(X)\rangle$, we have $\nabla H(X) = \mathcal{J}^{-1}(X)$, so that $\left.\frac{\mathsf{d}}{\mathsf{d}t}\right|_{t=0} \nabla H\left(\mathrm{Ad}_{e^{tB}} X_0\right) = \mathcal{J}^{-1}\left(\mathrm{ad}_B X_0\right)$, $B \in \mathfrak{u}(n)$. Under the assumption $\mathsf{J} = \mathrm{diag}\left(J_1, \dots, J_n\right)$, the action by $\mathcal{J}$ can be described as

$$\begin{aligned}
&\mathcal{J}\left(\mathsf{H}_i\right) = 2J_i\mathsf{H}_i, && 1 \le i \le n; \\
&\mathcal{J}\left(\mathsf{X}_{ij}\right) = \left(J_i + J_j\right)\mathsf{X}_{ij}, \quad \mathcal{J}\left(\mathsf{Y}_{ij}\right) = \left(J_i + J_j\right)\mathsf{Y}_{ij}, && 1 \le i < j \le n.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\mathcal{J}^{-1}(X_0) &= \sum_{i=1}^{n} \frac{x_i}{2J_i}\mathsf{H}_i, \\
\mathcal{J}^{-1}\left(\mathrm{ad}_{X_{ij}} X_0\right) &= -\frac{x_i - x_j}{J_i + J_j}\mathsf{Y}_{ij}, \\
\mathcal{J}^{-1}\left(\mathrm{ad}_{Y_{ij}} X_0\right) &= \frac{x_i - x_j}{J_i - J_j}\mathsf{X}_{ij},
\end{aligned}$$

where $1 \le i < j \le n$. Using these formulas, we calculate the nonzero components of the matrix representation of $\mathrm{Hess} H|_{\mathcal{O}}(X_0)$ as

$$\begin{aligned}
\left\langle \mathrm{ad}_{X_{ij}} X_0, \mathrm{Hess} H|_{\mathcal{O}}(X_0) \cdot \mathrm{ad}_{X_{ij}} X_0 \right\rangle &= \left\langle \mathrm{ad}_{X_{ij}} X_0, \mathcal{J}^{-1}\left(\mathrm{ad}_{X_{ij}} X_0\right)\right\rangle \\
&\quad - \sum_{k=1}^{n} \left\langle \mathcal{J}^{-1}(X_0), \mathsf{H}_k \right\rangle \left\langle \mathrm{ad}_{X_{ij}} X_0, \mathrm{ad}_{X_{ij}} \mathsf{H}_k \right\rangle \\
&= (x_i - x_j)\left\{ \left( \frac{1}{J_i + J_j} - \frac{1}{2J_i} \right) x_i - \left( \frac{1}{J_i + J_j} - \frac{1}{2J_j} \right) x_j \right\}
\end{aligned}$$

and

$$\begin{aligned}
\left\langle \mathrm{ad}_{Y_{ij}} X_0, \mathrm{Hess} H|_{\mathcal{O}}(X_0) \cdot \mathrm{ad}_{Y_{ij}} X_0 \right\rangle &= \left\langle \mathrm{ad}_{Y_{ij}} X_0, \mathcal{J}^{-1}\left(\mathrm{ad}_{Y_{ij}} X_0\right)\right\rangle \\
&\quad - \sum_{k=1}^{n} \left\langle \mathcal{J}^{-1}(X_0), \mathsf{H}_k \right\rangle \left\langle \mathrm{ad}_{Y_{ij}} X_0, \mathrm{ad}_{Y_{ij}} \mathsf{H}_k \right\rangle \\
&= (x_i - x_j)\left\{ \left( \frac{1}{J_i + J_j} - \frac{1}{2J_i} \right) x_i - \left( \frac{1}{J_i + J_j} - \frac{1}{2J_j} \right) x_j \right\},
\end{aligned}$$

where $1 \le i < j \le n$. The other components are zero, so that the matrix representation of the Hessian of $H|_{\mathcal{O}}$ at $X_0$ is given as the direct sum of the matrices

$$(x_i - x_j)\left\{ \left( \frac{1}{J_i + J_j} - \frac{1}{2J_i} \right) x_i - \left( \frac{1}{J_i + J_j} - \frac{1}{2J_j} \right) x_j \right\} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with respect to the basis $\mathrm{ad}_{X_{ij}} X_0$ and $\mathrm{ad}_{Y_{ij}} X_0$, $1 \le i < j \le n$. In other words, this basis diagonalizes the Hessian of $H|_{\mathcal{O}}$.

Therefore, we have the following proposition.

**Proposition 5.3.** *The linearization matrix of the Hamiltonian vector field of $H|_{\mathcal{O}}$ is the direct sum of $2 \times 2$ blocks with respect to the basis $\mathrm{ad}_{X_{ij}} X_0$ and $\mathrm{ad}_{Y_{ij}} X_0$, $1 \le i < j \le n$:*

$$\bigoplus_{1 \le i < j \le n} \left( \left\{ \left( \frac{1}{J_i + J_j} - \frac{1}{2J_i} \right) x_i - \left( \frac{1}{J_i + J_j} - \frac{1}{2J_j} \right) x_j \right\} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right),$$

*where the direct sum is that of the linear endomorphisms of* $\bigoplus_{1 \le i < j \le n} \mathrm{span}\left\{ \mathrm{ad}_{\mathsf{X}_{ij}} X_0, \mathrm{ad}_{\mathsf{Y}_{ij}} X_0 \right\} = T^*_{X_0}\mathcal{O}$.

From this result, we can see that the linearization matrix has only purely imaginary eigenvalues.

**Theorem 5.4.** *The equilibria $X_0 \in \mathfrak{h}_0 \cap \mathcal{O}$ on a generic orbit $\mathcal{O}$ are linearly stable.*

By Proposition 4.3 and Theorem 4.5, we conclude the nonlinear stability of the isolated equilibria on generic orbits.

**Theorem 5.5.** *Let $\mathcal{O}$ be a generic orbit consisting of invertible matrices with distinct eigenvalues. The equilibria $X_0 = \sum_{i=1}^{n} x_i \mathsf{H}_i \in \mathfrak{h}_0 \cap \mathcal{O}$ are Lyapunov stable, if all the elements in the set*

$$\left\{ \frac{x_j - x_i}{x_i J_j^2 - x_j J_i^2}, -\frac{1}{J_k^2} \;\middle|\; 1 \leq i < j \leq n,\, k = 1, \ldots, n \right\}$$

*are distinct.*

In comparison with the stability of the $SO(n)$ free rigid body dynamics, this result is remarkable, since in the case of $SO(n)$, $n \geq 3$, there are unstable equilibria on generic orbits.

**Remark 5.1.** The stability of isolated equilibria on a generic adjoint orbit is also deduced in [19], using another algebro-geometric method. The advantage of our method lies on the explicit calculation of the linearized Hamilton equations (Propositions 5.1 and 5.3), as well as the frequencies

$$\left( \frac{1}{J_i + J_j} - \frac{1}{2J_i} \right) x_i - \left( \frac{1}{J_i + J_j} - \frac{1}{2J_j} \right) x_j, \qquad 1 \leq i < j \leq n,$$

of the linearized $U(n)$ free rigid body (deduced from the formula in Proposition 5.3), which are useful in the study of the $U(n)$ free rigid body dynamics, including its perturbations. $\Diamond$

# 6 Example

Here, we explain an example of the $U(n)$ free rigid body in the case $n = 2$. For a skew-Hermitian matrix $X = \begin{bmatrix} \sqrt{-1}x_1 & z \\ -\bar{z} & \sqrt{-1}x_2 \end{bmatrix} \in \mathfrak{u}(2)$, $x_1, x_2 \in \mathbb{R}$, $z \in \mathbb{C}$, and for $\mathsf{J} = \operatorname{diag}(J_1, J_2)$, where $J_1$ and $J_2$ are distinct real numbers, we have $\mathcal{J}^{-1}(X) = \begin{bmatrix} \sqrt{-1}\dfrac{x_1}{2J_1} & \dfrac{z}{J_1 + J_2} \\ -\dfrac{\bar{z}}{J_1 + J_2} & \sqrt{-1}\dfrac{x_2}{2J_2} \end{bmatrix}$, so that the Euler equation for the $U(2)$ free rigid body is given as

$$\begin{cases} \dot{x}_1 &= 0, \\ \dot{x}_2 &= 0, \\ \dot{z} &= \dfrac{\sqrt{-1}(J_1 - J_2)}{2(J_1 + J_2)} \left( \dfrac{x_1}{J_1} + \dfrac{x_2}{J_2} \right) z, \end{cases} \tag{6.1}$$

since

$$[X, \mathcal{J}^{-1}(X)] = \begin{bmatrix} 0 & \dfrac{\sqrt{-1}(J_1 - J_2)}{2(J_1 + J_2)} \left( \dfrac{x_1}{J_1} + \dfrac{x_2}{J_2} \right) z \\ \dfrac{\sqrt{-1}(J_1 - J_2)}{2(J_1 + J_2)} \left( \dfrac{x_1}{J_1} + \dfrac{x_2}{J_2} \right) \bar{z} & 0 \end{bmatrix}.$$

The two functions

$$f_1(X) := -\sqrt{-1}\operatorname{Tr}(X) = x_1 + x_2,$$
$$f_2(X) := -\frac{1}{2}\operatorname{Tr}(X^2) = \frac{x_1^2 + x_2^2}{2} + |z|^2$$

are Casimir functions with respect to the Lie-Poisson bracket $\{\cdot,\cdot\}$ on $\mathfrak{u}(2)^*$. A generic orbit in $\mathfrak{u}(2) \cong \mathfrak{u}(2)^*$ can be described as

$$\begin{cases} x_1 + x_2 & = c_1, \\ \dfrac{x_1^2 + x_2^2}{2} + |z|^2 & = c_2, \end{cases}$$

which is nothing but a two-dimensional sphere. The coordinates of the equilibria on this orbit are given as $(x_1, x_2, z) = \left( \dfrac{c_1 \pm \sqrt{4c_2 - c_1^2}}{2}, \dfrac{c_1 \mp \sqrt{4c_2 - c_1^2}}{2}, 0 \right)$. This agrees with Theorem 4.2. From the above Euler equation (6.1), we can see that $x_1$ and $x_2$ are constant on each integral curve and that $z(t) = \exp\left( \dfrac{\sqrt{-1}\,(J_1 - J_2)}{2\,(J_1 + J_2)} \left( \dfrac{x_1}{J_1} + \dfrac{x_2}{J_2} \right) t \right) z_0$ gives the solution such that $z(0) = z_0$. It is clear that the integral curves are all closed and the two critical points are Lyapunov stable. This agrees with the result of Theorem 5.5.

The Hamiltonian of the $U(2)$ free rigid body dynamics can be written as

$$H(X) = \frac{1}{2} \left\langle X, \mathcal{J}^{-1}(X) \right\rangle = \frac{1}{2} \left( \frac{x_1^2}{2J_1} + \frac{x_2^2}{2J_2} + \frac{2|z|^2}{J_1 + J_2} \right).$$

Using $x_2 = c_1 - x_1$, we can rewrite the equation of the orbit as

$$\left( x_1 - \frac{c_1}{2} \right)^2 + |z|^2 = c_2 - \frac{c_1^2}{4},$$

while the Hamiltonian is given as

$$\begin{aligned} H(X) = \frac{1}{2} \Bigg[ & \left( \frac{1}{2J_1} + \frac{1}{2J_2} - \frac{2}{J_1 + J_2} \right) \left\{ x_1 - \frac{1}{2} \left( \frac{2}{J_1 + J_2} - \frac{1}{J_2} \right) \left( \frac{1}{2J_1} + \frac{1}{2J_2} - \frac{2}{J_1 + J_2} \right)^{-1} c_1 \right\}^2 \\ & + \left( \frac{1}{2J_2} - \frac{1}{J_1 + J_2} \right) c_1^2 + \frac{2c_2}{J_1 + J_2} \\ & - \frac{1}{4} \left( \frac{2}{J_1 + J_2} - \frac{1}{J_2} \right)^2 \left( \frac{1}{2J_1} + \frac{1}{2J_2} - \frac{2}{J_1 + J_2} \right)^{-1} \Bigg]. \end{aligned}$$

Thus, the flow of the $U(2)$ free rigid body on the two-dimensional sphere, as the orbit, can be regarded as the flow induced by a squared height function measured along the $x_1$-axis with a constant multiple.

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